Minimum SER Block Precoding and Equalization for Frequency-Selective Fading Channels

Lingyang Song, Rodrigo C. de Lamare, Are Hjørungnes, Alister G. Burr, and Manav R. Bhatnagar

Abstract

In this paper, we study the joint transmitter and receiver design problem over frequency-selective fading channels. Motivated by the conventional MMSE-DFE, we present a simple approximate ML decision feedback equalizer at the receiver. We then perform precoding in a downlink scenario at the transmitter side to minimize the analytical symbol error rate (SER) of the proposed equalization scheme. Simulation results indicate that the proposed precoder achieves substantial performance gains over previously reported techniques.

Index Terms

Frequency-selective fading channels, block transmission systems, Gaussian approximation, precoding.

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Lingyang Song is with Peking University, Beijing, China (e-mail: lingyang.song@pku.edu.cn).
Are Hjørungnes and Manav R. Bhatnagar are with UniK–University Graduate Center, University of Oslo, NO–2027, Norway (e-mail: arehj@unik.no; manav@unik.no).
Rodrigo C. de Lamare and Alister G. Burr are with the Department of Electronics, University of York, York, YO 10 5DD, UK (e-mail: rcdl500@ohm.york.ac.uk; alister@ohm.york.ac.uk).
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I. INTRODUCTION

There is a growing interest in transmission schemes that can exploit the channel state information (CSI) via some feedback channels [1] [2]. This can further enhance performance and capacity, when perfect CSI is made available at the transmitter side [2]. Hence, designing transmitters based on exact CSI is well motivated, which is of particular great interest in downlink transmission [1]–[13]. Transmitter design based on a capacity criterion has been studied in [3] [4] which specifies the achievable rate of reliable communication. In [5], transmitter designs obeying the water-filling principle are reported based on partial CSI. A simple transmit zero-forcing filter (TxZF) is proposed in [6]. Linear precoders that minimize the mean square error (MSE) are reported in [7]–[9] under a power constraint and maximum signal-to-noise ratio (SNR) precoder is proposed in [10], all of which can substantially outperform the TxZF. In [11], a unified approach is given by exploiting convex optimization tools. In [12] [13], non-linear precoders are discussed based on the minimum MSE approach using Tomlinson-Harashima scheme [14] [15].

In order to design an effective inter-symbol interference (ISI) mitigation scheme, in this paper, we present a joint minimum SER precoding and equalization algorithm to combat ISI in dispersive channels and in block transmission systems. We firstly present an approximate ML decision feedback equalizer (A-ML-DFE). The main ideas behind the proposed equalizer are to effectively utilize a slicing window, decision feedback equalization, and a Gaussian approximation [16], [17] to obtain near-optimal performance and to realize low complexity. Moreover, by assuming that the knowledge of CSI can be exploited at the transmitter side, we propose an iterative numerical technique to minimize the closed-form SER of A-ML-DFE with respect to a precoder matrix under a power constraint. The precoder is built up by using the derivative of complex matrices [18] [19]. The major advantage is that the performance can be further improved with very trivial complexity increase at the receiver end. In [20], linear precoding with minimum BER criterion was proposed by minimizing the pairwise error probability (PEP) for MIMO-OFDM system, instead of directly minimizing the analytical BER. In [21], a power loading algorithm for V-BLAST was presented and an optimal precoding vector was obtained to improve the system performance. However, unlike [21], in this paper, we aim to construct a precoding matrix with flexible size by taking into both amplitude and phase of every element into account.

The rest of the paper is organized as follows: In Section II, we introduce the system model.
The A-ML-DFE scheme, theoretical analysis, and complexity discussion are described in Section III. Precoder design is given in Section IV. Simulation results are shown in Section V. In Section VI, the main conclusions are drawn. Two derivations are given in the appendices.

**Notation**: Boldface upper-case letters denote matrices, boldface lower-case letters denote vectors, $C^{i \times j}$ and $R^{i \times j}$ denote the set of $i \times j$ complex and real matrices respectively, $(\cdot)^T$ stands for transpose, $(\cdot)^*$ denotes complex conjugate, $(\cdot)^H$ represents conjugate transpose, vec$(\cdot)$ stacks the columns of the argument matrix into a long column vector in chronological order, $I_i$ stands for an $i \times i$ identity matrix, $0_{p \times l}$ represents a matrix of all zero entries of size $p \times l$, $E$ is used for expectation, var is used for variance, and $\|x\|^2 = x^H x$.

## II. System Model

The frequency-selective channel can be modeled using a finite impulse response (FIR) filter

$$H(z) = \sum_{k=0}^{L-1} h_k z^{-k};$$

(1)
where $H(z)$ denotes the $z$–transform of the impulse response and the length of the FIR filter is $L$. In this paper, for simplicity, we only consider a single input single output wireless communications system with block transmissions and assume that the channel is time-invariant. A precoder $F$ is applied at the transmitter side before the signals are sent over the frequency-selective channels, $H$, which can be expressed as

$$F(z) = \sum_{k=0}^{L'-1} f_k z^{-k};$$

(2)
where the length of the filter is $L'$. The received signals can be written as

$$r_k = h_k * f_k * s_k + n_k,$$

(3)
where $*$ denotes discrete-time convolution, and $n_k$ denotes independent samples of a zero-mean complex Gaussian random variable. We set $E[|s_k|^2] = 1$, and thus, $\text{var}(n_k) = \sigma^2 = 1/\text{SNR}$. (3) can be further rewritten in a vector form as

$$\mathbf{r} = \mathbf{HF}s + \mathbf{n},$$

(4)
where the input signals $\mathbf{s} = [s_1, \ldots, s_Q]^T \in \mathbb{C}^{Q \times 1}$, the received signals $\mathbf{r} = [r_1, \ldots, r_{N+L-1}]^T \in \mathbb{C}^{(N+L-1) \times 1}$, $N = L' + Q - 1$, and $\mathbf{n} = [n_1, \ldots, n_{N+L-1}]^T \in \mathbb{C}^{(N+L-1) \times 1}$. For simplicity, we
assume $s_k = 0, k \notin \{1, 2, \ldots, Q\}$. The time-domain presentation of $H \in \mathbb{C}^{(N+L−1)\times N}$, can be written as the following Toeplitz form

$$H = \begin{bmatrix}
h_0 & 0 & 0 & \cdots & 0 \\
h_1 & h_0 & 0 & \cdots & \vdots \\
\vdots & h_1 & h_0 & \cdots & 0 \\
h_{L−1} & \vdots & h_1 & \cdots & h_0 \\
0 & h_{L−1} & \vdots & \cdots & h_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & h_{L−1}
\end{bmatrix}.$$  

In order to make the transmit power unchanged, a power constraint must be applied as [1]

$$\mathbb{Tr}(FF^H) = Q, \quad (5)$$

where $F \in \mathbb{C}^{N\times Q}$ also has a Toeplitz form

$$F = \begin{bmatrix}
f_0 & 0 & 0 & \cdots & 0 \\
f_1 & f_0 & 0 & \cdots & \vdots \\
\vdots & f_1 & f_0 & \cdots & 0 \\
f_{L′−1} & \vdots & f_1 & \cdots & f_0 \\
0 & f_{L′−1} & \vdots & \cdots & f_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \vdots
\end{bmatrix}.$$  

It should be noted that some form of guard interval is necessary to avoid inter-block interference between the received signals, which must have larger length than the maximum channel delays. As the precoder is to minimize the SER of the A-ML-DFE, we describe the receiver design and derive its corresponding SER expression in the next section.

**III. DESCRIPTION OF THE PROPOSED RECEIVER DESIGN**

**A. Approximate Maximum Likelihood Decision Feedback Equalizer (A-ML-DFE)**

Let $C \triangleq HF \in \mathbb{C}^{(N+L−1)\times Q}$. Now, we have all the preliminaries, and, next, we will introduce the approximate ML decision feedback equalizer in the following three steps:
1) **Forward Process:** In this step, we first select $L_f (L \leq L_f < Q)$ consecutive scalar received signals from the vector $r$ in (4). Supposing we start decoding $s_k$ based on the $L_f$ received signals, we obtain the following expression

$$r_{k,k+L_f-1} = C_{k,k+L_f-1}^{k,k+L_f-1} s + n_{k,k+L_f-1},$$  

(6)

where $r_{k,k+L_f-1} = [r_k, \ldots, r_{k+L_f-1}]^T$, and $C_{k,k+L_f-1}^{k,k+L_f-1} \in \mathbb{C}^{L_f \times (p-1)+1}$, which is made up by the entries in $C$ from the $k$-th row to the $(k + L_f - 1)$-th row and from the $l$-th column to the $p$-th column. We select $l = 1$ and $p = Q$ in (6), and $n_{k,k+L_f-1} = [n_k, \ldots, n_{k+L_f-1}]^T$. We call this process **horizontal slicing**, since it takes $L_f$ rows of $H$. In order to further decrease the complexity of (6), we can just consider a certain number of the transmitted symbols, and have

$$r_{k,k+L_f-1} \approx \sum_{i=1}^{k-1} c_i s_i + c_k s_k + \sum_{i=k+1}^{k+L_f-1} c_i s_i + n_{k,k+L_f-1},$$

(7)

where $c_i$ stands for the $i$-th column of the matrix $H_{1,N}^{k,k+L_f-1}F_{k,k+L_f-1}$ with size $L_f \times 1$, $H_{l,p}^{k,k+L_f-1}$ has a similar structure to $C_{l,p}^{k,k+L_f-1}$, $F_{k,k+L_f-1} = [f_k, \ldots, f_{k+L_f-1}]$ has size $N \times L_f$, and $f_i$ represents the $i$-th column of $F$ with size $N \times 1$. We call this process **vertical slicing**, since it takes $k + L_f - 1$ columns of $C$.

2) **Decision Feedback Process:** The function of the feedback filter is to reconstruct $\sum_{i=1}^{k-1} c_i s_i$ for later interference cancellation. Therefore, it is important to decide the length of the backward filter, $L_b$. For simplicity, we only consider the worst case when no precoder is applied: $F = I_N$ ($C = H$). Since there are zero-valued elements in $H$ ($H_{k,i} = 0, i \leq k - L$), where $H_{k,i}$ represents the entry in $H$ at the $k$-th row and the $i$-th column, the length of the backward filter $L_b$ can be fixed at $L - 1, (L > 1)$ to reconstruct the effects of past decisions. The rest of the past decisions have no effect at all since they correspond to zero elements in $H$, if we design the filter properly. Hence, we can get

$$\tilde{r}_{k,k+L_f-1} = r_{k,k+L_f-1} - \sum_{i=k-L_b}^{k-1} c_i s_i = c_k s_k + \sum_{i=k+1}^{k+L_f-1} c_i s_i + n_{k,k+L_f-1},$$

(8)

where $L_b$ equals $L - 1$.

One improvement compared to MMSE-DFE is that we do not need to calculate the coefficients of the feedback filter, and what is more, the length of the $L_b$ is fixed at $L - 1$, which means only $L - 1$ past decisions need to be fed back, which is much less than what is typically required by the MMSE-DFE.
3) Approximate ML: In order to decode $s_k$ with low computational complexity and meanwhile maintain its performance compared to the ML decoder, we treat the undetected terms $\sum_{i=k+1}^{L_f-1}c_i s_i$ and the noise vector $n_{k,k+L_f-1}$ in (8) together as an approximate complex-valued Gaussian vector with matching mean and covariance matrix $\Lambda_k = C_{k,k+L_f-1}^{k+1,k+L_f-1} + \sigma^2 I_{L_f}$. After applying the pre-whitening filter, $\Theta_k = \Lambda_k^{-\frac{1}{2}}$, and the matched filter, $(\Theta_k c_k)^H$, (8) can be further expressed in a scalar form

$$y_{k,k+L_f-1} = \xi_k s_k + v_k,$$

where $y_{k,k+L_f-1} = (\Theta_k c_k)^H \Theta_k r_{k,k+L_f-1} = c_k^H \Lambda_k^{-1} r_{k,k+L_f-1}$, $\xi_k = \|\Theta_k c_k\|^2 = c_k^H \Lambda_k^{-1} c_k$, and $v_k$ is a Gaussian scalar with zero mean and variance $\|\Theta_k c_k\|^2$.

Hence, all the possible modulated symbols related to $s_k$ can be examined by the following ML detector

$$\tilde{s}_k = \arg\min_{s_k \in \mathcal{A}} \left| y_{k,k+L_f-1} - \xi_k s_k \right|^2,$$

where $\mathcal{A}$ represents the signal constellation.

B. Theoretical Analysis

In this subsection, we analyze the performance of the A-ML-DFE in term of the SER. We assume that all the decisions are accurate, which is a normal assumption in decision feedback theory [22]. For $M$-PSK constellations, the SER for the $k$-th symbol, $s_k$, is given by [23]

$$\text{SER}_M^k = \frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} \exp \left( -\frac{g_{\text{PSK}}\gamma_k}{\sin^2 \theta} \right) d\theta,$$

where $\gamma_k \triangleq \frac{\|c_k\|^2}{\text{var}(c_k)} = \frac{\xi_k^2}{(\Theta_k c_k)^H (\Theta_k c_k)} = \|\Theta_k c_k\|^2$, and $g_{\text{PSK}} \triangleq \sin^2 \frac{\pi}{M}$. The SER for $M$-QAM of $s_k$ can be also found in [23]

$$\text{SER}_M^k = \frac{4}{\pi} \left( 1 - \frac{1}{\sqrt{M}} \right) \left[ \frac{1}{\sqrt{M}} \int_0^{\frac{\pi}{2}} \exp \left( -\frac{g_{\text{QAM}}\gamma_k}{\sin^2 \theta} \right) d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \exp \left( -\frac{g_{\text{QAM}}\gamma_k}{\sin^2 \theta} \right) d\theta \right],$$

where $g_{\text{QAM}} \triangleq \frac{3}{2(M-1)}$. Then, the average closed-form SER over $Q$ transmit symbols can be calculated as

$$\text{SER} = \frac{1}{Q} \sum_{k=1}^{Q} \text{SER}_M^k.$$

The average BER for $M$-PSK or $M$-QAM can be written as:

$$\text{BER} = \frac{1}{Q} \sum_{k=1}^{Q} \text{BER}_M^k,$$

where $\text{BER}_M^k \approx \frac{1}{\log_2 M} \text{SER}_M^k$ [24] for high SNR and Gray mapping.
Next, for simplicity and without loss of generality, we analyze further the behavior of the proposed A-ML-DFE for high SNR values with $M$-QAM constellations. Assuming perfect channel state information at the receiver, and taking (12) as an example, it can be upper bounded by [24]

$$\text{SER}_M^k \leq 2\exp \left( -\frac{3 \cdot \log_2 M}{2(M-1)} \cdot \gamma_k \right) \leq 2\exp \left( -\frac{3 \cdot \log_2 M}{2(M-1) \cdot \sigma^2} \cdot L_f \sum_{i=0}^{L-1} |h_i|^2 \right),$$

where $h_i$ is i.i.d. circularly symmetric complex Gaussian. Refer to Appendix I for the derivation of (15). By averaging (15) over the Rayleigh PDF [25], we finally obtain

$$\text{SER}_M^k \leq 2 \left( \frac{3 \cdot \log_2 M \cdot \text{SNR}}{(M-1) \cdot L_f} \right)^L,$$

which shows that the mean BER is approximately inversely proportional to the SNR to the power of $L$. This demonstrates that $L$ will determine the slope of the BER curve, which also implies that A-ML-DFE is able to obtain at least the same diversity order as the MLSE decoder.

C. Computational Complexity Discussion

In this subsection, we present the complexity of the A-ML-DFE, linear-MMSE [22], and MMSE-DFE [22] in terms of the number of additions and multiplications. The main focus of this paper is to find $F$ which minimizes the SER of the A-ML-DFE to combat ISI in dispersive fading channels. Note that with precoding, $HF$ can be treated as the channel matrix, which can be obtained by performing channel estimation at the receiver side. Since $HF$ is still Toeplitz, the computation complexity of channel estimation changes very slightly when precoder is applied. Hence, with additional precoder, the total complexity increase is negligible and similar to the A-ML-DFE scheme. For fairness, we assume there is no precoding ($F = I$) for any complexity comparison.

The resulting values are given in Table II, obtained by inspection of the relevant algorithms in Table I and [22]. Details of the computation of complexity, for example the matrix inversion, can be found in [26]. The computational complexity of the A-ML-DFE algorithm is a function of the length of frame ($N$), the impulse response length ($L$), and the length of the forward filter ($L_f$). From the table, we observe that A-ML-DFE has the same order of complexity as Linear-MMSE and MMSE-DFE. But A-ML-DFE is less complex than MMSE-DFE since A-ML-DFE requires a smaller forward filter length, and it does not require to build up the backward filter. In comparison to Linear-MMSE, A-ML-DFE needs
relatively even shorter forward filter and thus, has lower complexity. Note that with regard to computational complexity, we focus on time-domain implementation even though a low-complexity frequency-domain implementation is also possible by making use of the block-circulant structure that can be created by the guard interval.

IV. Precoder Design

The precoder design corresponds to finding a matrix $F$ such that the SER in (12) is minimized subject to the power constraint, by solving the following optimization problem

$$\mathcal{L}(F) = \sum_{k=1}^{Q} \text{SER}_M^k + \mu \text{Tr}(FF^H),$$

where $\mu$ is a positive Lagrangian multiplier. The necessary condition for the optimality of the precoder construction can be found by setting the derivative of the Lagrangian in (17) with respect to $\text{vec}(F^*) \in \mathbb{C}^{NQ \times 1}$ equal to zero. The following two expressions, which are found after several matrix manipulations, are useful:

$$D_{F^*} \text{Tr}(FF^H) = \text{vec}^T(F),$$

and

$$D_{F^*} \text{SER}_M^k = -\frac{4 \cdot g_{qam}}{\pi} \left(1 - \frac{1}{\sqrt{L}}\right) \cdot (D_{F^*}\gamma_k) \cdot \left[ \frac{1}{\sqrt{L}} \int_0^{\frac{\pi}{2}} e^{-\frac{g_{qam}\gamma_k}{\sin^2 \theta}} \sin^2 \theta d\theta + \int_0^{\frac{\pi}{2}} e^{-\frac{g_{qam}\gamma_k}{\sin^2 \theta}} \sin^2 \theta d\theta \right],$$

where

$$D_{F^*}\gamma_k = (P^H \Lambda^{-1}_{k} Pf_k^*)^T [0_{N \times (k-1)N}, I_N, 0_{N \times (Q-k)N}]$$

$$- \left[ (f_k^TP^H \Lambda^{-1}_{k} P^*) \otimes (f_k^H P^H \Lambda^{-1}_{k} Pf_{k+1,k+L_f-1}) \right] K_{N,L_f}$$

$$\times \left[ 0_{N(L_f-1)\times Nk}; I_{N(L_f-1)}; 0_{N(L_f-1)\times N(Q-k+1,L_f)} \right].$$

$P \triangleq H_{1,N}^{k,k+L_f-1}$, $\otimes$ represents the Kronecker product, and $K_{q,w} \in \mathbb{R}^{qw \times qw}$ is the commutation matrix [19]. Refer to Appendix II for the derivation of (20).

By utilizing the results from (18) and (20) and setting the derivative of the Lagrangian in (17) equal to zero, the precoder can be found in an iterative way

$$\text{vec}^T(F_{i+1}) = -\frac{1}{\mu_i} \sum_{k=1}^{Q} D_{F_i^*} \text{SER}_M^k$$

where $F_i$ and $\mu_i$ represents the precoder matrix and the Lagrangian multiplier at the $i$-th iteration respectively. Using the power constraint, $\mu_i$ can be obtained by solving
\[ \text{Tr}(\mathbf{F}_{i+1}^H \mathbf{F}_{i+1}) = \text{vec}^H(\mathbf{F}_{i+1}) \text{vec}(\mathbf{F}_{i+1}) = Q, \]  

(22)

where \( \mu_i > 0 \).

This algorithm is guaranteed to converge at least to a local minimum [27], since at each step the objective function is decreased and is bounded below by zero. In the numerical experiments, we observed that the proposed fixed-point algorithm always converged. We have experimented with many different initialization matrices and never experienced any divergence of the proposed fixed-point algorithm. The initial value of the precoder can be set as \( \mathbf{F}_0 \propto \mathbf{I} \). The overall precoded A-ML-DFE algorithm is summarized in Table I.

V. SIMULATION RESULTS

In all simulations, BPSK and QPSK constellations are used to generate a rate 1bps/Hz transmission. We plot the symbol error rate (SER) versus the signal-to-noise ratio (SNR). The channel is frequency-selective but time-invariant, and is assumed to be perfectly estimated at both the transmitter and the receiver side. Performance is determined over frequency-selective fading channels which remain time-invariant during one frame length and vary from one frame to another. The impulse response length is five \( L = 5 \), and, thus, the length of the backward filter of the A-ML-DFE can be fixed as \( L_b = L - 1 = 4 \). The length of the frame, \( Q = 128 \). The length of the precoder is \( L' = L \), and thus, \( N = Q + L - 1 = 132 \). For analytical results, we assume perfect decision feedback, but for simulated results we use the feedback decisions. In this subsection, we examine the BER performance of the proposed A-ML-DFE, without pre-processing at the transmitter side.

Here, we evaluate the analytical and simulated performance of the precoded A-ML-DFE system, as described in Section IV. In Fig. 1, we evaluate the simulated and theoretical performance of a system which uses precoding. We set \( L_f = 15 \) for all the schemes. It shows that there is about 0.8 dB performance gap at \( \text{SER}=10^{-5} \) between the A-ML-DFE with \( L_f = 15 \) and MLSE. It can be seen that the use of a precoder can improve the system performance significantly in comparison with the case in which no transmitter design is used in the whole SNR region. Around 3 dB improvement can be observed at \( \text{BER}=10^{-5} \). The analytical SER of the precoder is also close and asymptotically converges to the simulated curves at high SNR. We can also see that as the transmitter has perfect CSI, the minimum SER precoder provides better performance than MLSE.

In Fig. 2, we compare the SER of the A-ML-DFE system employing transmitter design with various other precoding schemes using BPSK constellations. The proposed precoder
can improve the system performance in the entire SNR regime by minimizing the analytical SER. We can observe that the proposed A-ML-DFE without any transmitter design is able to provide much better performance than ZF based precoder [6], Linear MMSE and MMSE DFE precoders in [7]. Hence, much larger gain can be achieved by the precoded A-ML-DFE scheme. We also include the power-allocation based minimum SER precoder in [21] for comparison, it shows that our proposed scheme obtains better performance. The main reason is that the proposed design method considers not only the amplitude but also the phase of every element in the precoding matrix.

In Fig. 3, we compare the SER of the precoded A-ML-DFE system with various other precoding schemes using QPSK constellations. We have observed that the proposed A-ML-DFE precoder obviously outperforms ZF based precoder [6], Linear MMSE and MMSE DFE precoders in [7], and power-allocation based minimum SER precoder in [21].

In Fig. 4, we examine the convergence behavior in (21) of the proposed A-ML-DFE precoder for SNR=8 dB using BPSK and QPSK constellations, respectively. It is seen from the plot that the algorithm converges very fast. Typically, within one to two iterations the algorithm has converged to a satisfactory performing precoder. In addition, by comparing the results using BPSK and QPSK constellations, it shows from Fig. 4 that the type of constellations does not affect the convergence speed.

VI. CONCLUSIONS

In this paper, we have presented an A-ML-DFE based precoder for frequency selective fading channel. The precoder is constructed by minimizing the analytical SER of the proposed equalizer. Simulation results show that the proposed schemes can provide a performance much better than the existing schemes with a low complexity. Note that we assume perfect channel knowledge at the transmitter side, however, due to fast or deep fading effects, channel state information on the uplink can be inaccurate. In this case, the transmitter has to be jointly designed with the condition of the channel feedback [28]–[30]. This problem will be considered in our future work.
**Appendix I**

**Derivation of (15)**

Here, we only consider the worst case that no precoder is used, \( F = I \) \((C = H)\). Recalling (9), we can obtain

\[
\gamma_k = \|\Theta_k h_k\|^2 = h_k^H \Lambda_k^{-1} h_k = h_k^H (DD^H + \sigma^2 I_{L_f})^{-1} h_k, \tag{23}
\]

where \( h_k \) stands for the \( k \)-th column of the matrix \( H_{k,k+L_f-1} \), and for convenience we define

\[
\mathbf{D} \triangleq H_{k,k+L_f-1}^{k+1,k+L_f-1}.
\]

Using the Kailath Variant [31]

\[
(A + BC)^{-1} = A^{-1} - A^{-1} B (I + CA^{-1} B)^{-1} CA^{-1},
\]

we have

\[
\gamma_k = \sigma^{-2} h_k^H (I_{L_f} - D(D^H D + \sigma^2 I_{L_f-1})^{-1} D^H) h_k. \tag{24}
\]

At high SNR, \( \sigma^2 \to 0 \), we can get

\[
\gamma_k = \sigma^{-2} h_k^H (I_{L_f} - D(D^H D)^{-1} D^H) h_k. \tag{25}
\]

Let \( \mathbf{D}^+ \) be the Moore-Penrose inverse of matrix \( \mathbf{D} \) [32], and we can get

\[
\mathbf{D}^+ = (D^H D)^{-1} D^H, \tag{26}
\]

which has size \((L_f - 1) \times L_f\), and thus \( \mathbf{D} \mathbf{D}^+ \) has size \( L_f \times L_f \). Note that \( \text{rank}(\mathbf{D} \mathbf{D}^+) = \text{rank}(H_{H_{k+1},N}^H H_{k+1,N}) = L_f - 1 \). It can be shown that

\[
(\mathbf{D} \mathbf{D}^+)^H = \mathbf{D} \mathbf{D}^+, \quad (\mathbf{D} \mathbf{D}^+)^H \mathbf{D} \mathbf{D}^+ = \mathbf{D} \mathbf{D}^+. \tag{27}
\]

By eigenvalue decomposition, further transformation can be carried out

\[
I_{L_f} - \mathbf{D} \mathbf{D}^+ = \mathbf{U} (I_{L_f} - \boldsymbol{\Pi}) \mathbf{U}^H, \tag{28}
\]

where \( \mathbf{U} \) is a unitary matrix, \( \boldsymbol{\Pi} \triangleq \text{diag}\{\lambda_1, \ldots, \lambda_{L_f-1}, 0\} \) with size \( L_f \times L_f \), and \( \mathbf{D} \mathbf{D}^+ = \mathbf{U} \boldsymbol{\Pi} \mathbf{U}^H \). Then, using (27), we can find

\[
(I_{L_f} - \mathbf{D} \mathbf{D}^+)^2 = I_{L_f} - \mathbf{D} \mathbf{D}^+. \tag{29}
\]

Therefore, \( I_{L_f} - \mathbf{D} \mathbf{D}^+ \) is idempotent, and any idempotent matrix has eigenvalue 1 or 0 [33]. It follows that

\[
I_{L_f} - \boldsymbol{\Pi} = \text{diag}\{0, \ldots, 0, 1\}. \]

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Hence, (23) can be rewritten as

\[ \gamma_k = \| \Theta_k h_k \|^2 \]
\[ = \sigma^{-2} h_k^H (I_{L_f} - DD^+) h_k \]
\[ = \sigma^{-2} h_k^H U (I_{L_f} - \Pi) U^H h_k \]
\[ \approx \frac{1}{\sigma^2 L_f} \| h_k^H U \|^2 \]
\[ = \frac{1}{\sigma^2 L_f} \| h_k \|^2. \]

where the length of \( L_f \) should be at least \( L \) in order to contain all the channel elements for maximum multipath diversity order. Finally, we can obtain

\[ \gamma_k \approx \frac{1}{L_f} \sum_{i=0}^{L-1} |h_i|^2. \] (31)

**APPENDIX II**

**DERIVATION OF (20)**

For convenience, let \( P \triangleq H_{1:N,1:N}^{k+L_f-1}. \) Since \( \gamma_k = f_k^H P^H \Lambda_k^{-1} F_k \), we can get

\[ d\gamma_k = f_k^H P^H \Lambda_k^{-1} P (dF_k) + (dF_k^H) P^H \Lambda_k^{-1} F_k + f_k^H P^H (d\Lambda_k^{-1}) F_k. \] (32)

Note that the derivatives are with respect to \( \text{vec}(F^*) \), and thus, the terms in (32) containing \( df_k \) (or \( dF_{k+1,k+L_f-1} \) in (35)) will not be considered in the following derivation process.

1) Since \( \gamma_k \) is a scalar, we can get

\[ (dF_k^H) P^H \Lambda_k^{-1} F_k = (P^H \Lambda_k^{-1} F_k)^T dF_k^*. \] (33)

As \( \text{vec}(f_k^*) = [0_{N \times (k-1)N}, I_N, 0_{N \times (Q-k)N}] \text{vec}(F^*) \), it yields that

\[ (dF_k^H) P^H \Lambda_k^{-1} F_k = (P^H \Lambda_k^{-1} F_k)^T [0_{N \times (k-1)N}, I_N, 0_{N \times (Q-k)N}] d\text{vec}(F^*). \] (34)

2) It is known that \( d\Lambda_k^{-1} = -\Lambda_k^{-1} (d\Lambda_k) \Lambda_k^{-1} \) [19]. Recalling Section III-A, we have \( \Lambda_k = PF_{k+1,k+L_f} F_{k+1,k+L_f}^H + \sigma^2 I_{L_f} \) and thus

\[ d\Lambda_k = PF_{k+1,k+L_f-1} (dF_{k+1,k+L_f-1}^H) P^H + P (dF_{k+1,k+L_f-1}) F_{k+1,k+L_f-1}^H P^H. \] (35)

Then, the third term in (32) can be written as
\[
\mathbf{f}_k^H \mathbf{P}^H (d\Lambda_k^{-1}) \mathbf{P} \mathbf{f}_k = -\mathbf{f}_k^H \mathbf{P}^H \Lambda_k^{-1} (d\Lambda_k) \Lambda_k^{-1} \mathbf{P} \mathbf{f}_k \\
= -\mathbf{f}_k^H \mathbf{P}^H \Lambda_k^{-1} \mathbf{P} \mathbf{F}_{k+1,k+L_f-1} \left( d\mathbf{F}_k^H \right) \mathbf{P}^H \Lambda_k^{-1} \mathbf{P} \mathbf{f}_k \\
- \mathbf{f}_k^H \mathbf{P}^H \Lambda_k^{-1} \mathbf{P} \left( d\mathbf{F}_k \right) \mathbf{F}_{k+1,k+L_f-1} \mathbf{P}^H \Lambda_k^{-1} \mathbf{P} \mathbf{f}_k. \tag{36}
\]

Note that \( \text{vec}(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A}) \text{vec}(\mathbf{B}) \) can be used to find the \( \text{vec}(\cdot) \) of an inner matrix from the \( \text{vec}(\cdot) \) of a multiple matrix product [32]. Since \( \gamma_k \) is a scalar and only the term in (36) containing \( d\mathbf{F}_k \) will be considered, we can get

\[
-\mathbf{f}_k^H \mathbf{P}^H \Lambda_k^{-1} \mathbf{P} \mathbf{F}_{k+1,k+L_f-1} \left( d\mathbf{F}_k \right) \mathbf{P}^H \Lambda_k^{-1} \mathbf{P} \mathbf{f}_k \\
= - \left[ (\mathbf{f}_k^T \mathbf{P}^H \Lambda_k^{-T} \mathbf{P}^*) \otimes (\mathbf{f}_k^H \mathbf{P}^H \Lambda_k^{-1} \mathbf{P} \mathbf{F}_{k+1,k+L_f-1}) \right] d\text{vec}(\mathbf{F}_k^H) \tag{37}
\]

For the term \( \text{vec}(\mathbf{F}_k^H) \) in (37), the commutation matrix \( \mathbf{K}_{q,w} \in \mathbb{C}^{qw \times qw} \) defines the connection between \( \text{vec}(\mathbf{X}^H) \) and \( \text{vec}(\mathbf{X}^*) \) in the following way: \( \text{vec}(\mathbf{X}^H) = \mathbf{K}_{q,w} \text{vec}(\mathbf{X}^*) \) [19], where \( \mathbf{X} \in \mathbb{C}^{q \times w} \). Hence, we can get

\[
\text{vec}(\mathbf{F}_k^H) = \mathbf{K}_{N,L_f-1} \text{vec}(\mathbf{F}_k^{*}) \tag{38}
\]

We can also have

\[
\text{vec}(\mathbf{F}_k^{*}) = \left[ \mathbf{0}_{N(L_f-1) \times Nk}, \mathbf{I}_{N(L_f-1)}, \mathbf{0}_{N(L_f-1) \times (Q-k+1+L_f)} \right] \text{vec}(\mathbf{F}^*). \tag{39}
\]

The derivatives are with respect to \( \text{vec}(\mathbf{F}^*) \), and thus, the terms in (32) containing \( df_k \) or \( df_{k+1,k+L_f-1} \) are not going to be used. By combining (34) and (37)–(39), we can finally obtain (20).

**REFERENCES**


TABLE I
PRECODED APPROXIMATE MAXIMUM LIKELIHOOD DECISION FEEDBACK EQUALIZATION ALGORITHM.

Step 1: Precoder Design
1. Find $\mathbf{F}$ using the following iterative method:
   
   $\mathbf{F}_0 \propto \mathbf{I}$;
   $i = 0$;
   repeat
   
   $i := i + 1$;
   Calculate $\mathbf{F}_{i+1}$ using (21);
   until $\| \vec{\mathbf{F}}_{i+1} - \vec{\mathbf{F}}_i \| < \epsilon$;
   $\mathbf{F} = \mathbf{F}_i$;
2. Or set $\mathbf{F} \propto \mathbf{I}$ if no precoder is needed.

Step 2: Initialization of A-ML-DFE
1. Set $L_f (L_f \geq L)$.
2. Fix $L_b = L - 1$.

Step 3: A-ML-DFE Detection
For $k = 1 : Q$
1. Forward Process: Apply horizontal slicing by selecting consecutive $L_f$ received signals according to (6), and vertical slicing of length $k + L_f - 1$ according to (7).
2. Decision Feedback Process: Remove the effects reconstructed by the past decisions using (8).
3. Approximate ML: Recover the transmitted signals with (10).

end
TABLE II

**Computational complexity of various schemes for one sliding window with length N; L is the number of paths; BPSK constellations; \( L_f \) and \( L_b \) stand for the length of the forward and backward filters correspondingly, \( F = I_N \).**

<table>
<thead>
<tr>
<th>Detector</th>
<th>Additions</th>
<th>Multiplications</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-ML-DFE</td>
<td>( N[8L_f^3 + 34L_f^2 + (6L + 7)L_f + (3L - 1)] )</td>
<td>( N[(2L_f^3 + 42L_f^2) - (12L + 19)L_f + 18] )</td>
</tr>
<tr>
<td>Linear-MMSE</td>
<td>( N[8L_f^3 + 30L_f^2 + 2(3L + 2)L_f] )</td>
<td>( N[2L_f^3 + 42L_f^2 - (12L - 17)L_f - (6L - 1)] )</td>
</tr>
<tr>
<td>MMSE-DFE</td>
<td>( N[8(L_f^3 + L_b^3) + 42(L_f^2 + L_b^2) + 2(3L + 2)(L_f + L_b)] )</td>
<td>( N[2(L_f^3 + L_b^3) + 42(L_f^2 + L_b^2) + (12L - 11)(L_f + L_b) + 6] )</td>
</tr>
</tbody>
</table>
Fig. 1. SER Performance of the A-ML-DFE and the precoded A-ML-DFE over frequency-selective fading channels (BPSK, $L = 5$, $Q = 128$, $L_f = 15$, and $L_b = L - 1 = 4$). Shown for comparisons are the A-ML-DFE, MLSE, and precoding results.
Fig. 2. SER Performance of the A-ML-DFE and the precoded A-ML-DFE over frequency-selective fading channels (BPSK, $L = 5$, $Q = 128$, $L_f = 15$, and $L_b = L - 1 = 4$). Shown for comparisons are the A-ML-DFE, ZF based Linear MMSE precoder ($L_f = 15$), minimum MSE based linear MMSE ($L_f = 15$), linear MMSE DFE precoders ($L_f = 15$, $L_b = 9$), MLSE, and power-allocation precoder.
Fig. 3. SER Performance of the precoded A-ML-DFE over frequency-selective fading channels (QPSK, $L = 5$, $Q = 128$, $L_f = 15$, and $L_b = L - 1 = 4$). Shown for comparisons are the A-ML-DFE, ZF based Linear MMSE precoder ($L_f = 15$), minimum MSE based linear MMSE ($L_f = 15$), linear MMSE DFE precoders ($L_f = 15$, $L_b = 9$), and power-allocation precoder.
Fig. 4. Empirical convergence of the precoded A-ML-DFE in (21) using BPSK and QPSK, respectively. Zero iteration corresponds to the system without employing a precoder (SNR = 8 dB, $L = 5$, $Q = 128$, $L_f = 15$, and $L_b = L - 1 = 4$).