

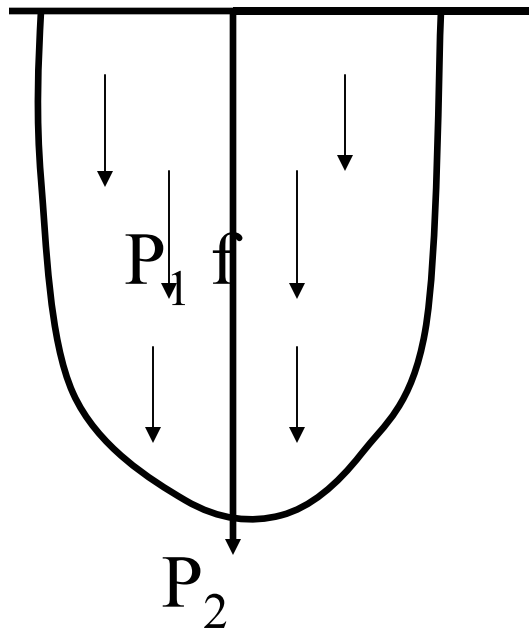
1D ELEMENTS

- Simplest type of FE problems
- All object are 1D
- All forces are 1D
- All stresses / strains are 1D

BASICS

- $u=u(x)$: Deformations
- $\varepsilon=\varepsilon(x)$: Strain
- $\sigma=\sigma(x)$: Stress
- $f=f(x)$: Body forces
- $T=T(x)$: Tractive forces
- $P=P(x)$: Point loads
- $\sigma=E\varepsilon, \varepsilon=du/dx$

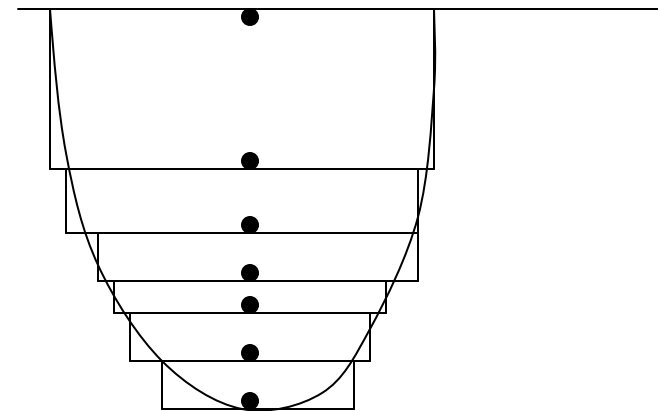
Problem



Discritization

Element connectivity Matrix

Elem No.	Node 1	Node 2
1	1	2
2	2	3
3	3	4
4	4	5
5	5	6
6	6	7



INDIVIDUAL ELEMENTS



\mathbf{x}_1

$\xi=-1$

q_1

\mathbf{x}_2

$\xi=1$

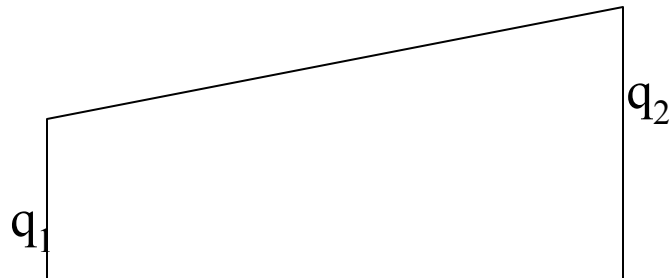
q_2

:Coordinates

: Local coordinates

: Deformations

(At the nodes)



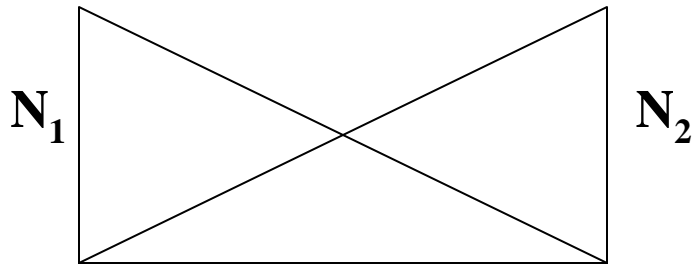
:Deformation

within the element

Linear Interpolation

N_1, N_2 : Shape functions

LINEAR SHAPE FUNCTIONS



$$N_1 = (1 - \xi) / 2$$

$$N_2 = (1 + \xi) / 2$$

$$u = N_1 q_1 + N_2 q_2 = (1 - \xi) / 2 * q_1 + (1 + \xi) / 2 * q_2$$

Relationship between local and global coordinates-

$$\xi = -1 + 2 * (x - x_1) / (x_2 - x_1)$$

STRAIN:

$$\varepsilon = du/dx$$

$$= du/d\xi * d\xi/dx$$

$$u = N_1 q_1 + N_2 q_2$$

$$N_1 = (1 - \xi) / 2$$

$$N_2 = (1 + \xi) / 2$$

$$d\xi/dx = 2/(x_2 - x_1)$$

$$= 2/L_e$$

$$du/d\xi = (-q_1 + q_2)/2$$

therefore-

$$\varepsilon = 1/l_e [-1 \ 1][q_1 \ q_2]^T$$

$$= Bq$$

Where B is the element strain matrix and

$$B = 1/l_e [-1 \ 1]$$

As B is constant, this element is CONSTANT STRAIN ELEMENT

That means strain inside the element does not vary.

STRESS:

$$\sigma = EBq$$

P.E. approach

$$\Pi = \frac{1}{2} \int_L \sigma^T \epsilon A dx - \int_L u^T F A dx - \int_L u^T T dx - \sum u_i P_i$$

$$\sum_e \frac{1}{2} \int_L \sigma^T \epsilon A dx - \sum_e \int_L u^T F A dx - \sum_e \int_L u^T T dx - \sum Q_i P_i$$



STRAIN ENERGY
FIELD



FORCED
FORCE



TRACTIVE
FORCE



POINT
LOADS

Total energy of the body is the sum of P.E, of all Elements.

ELEMENT STRAIN ENERGY

$$\begin{aligned}U_e &= \frac{1}{2} \int_L \sigma^T \epsilon A dx \\&= \frac{1}{2} \int_L q^T B^T E B q A dx \\&= \frac{1}{2} E A q^T \left[\int_L B^T B dx \right] q \\&= \frac{1}{2} E A q^T \cdot \left[\int_L \frac{1}{l_e^2} [-1 \quad 1]^T [-1 \quad 1] dx \right] q \\&= \frac{1}{2} E A q^T \frac{1}{l_e^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int dx \quad q \\&= \frac{1}{2} \frac{E A}{l_e} q^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} q\end{aligned}$$

$$U_e = \frac{1}{2} q^T k_e q$$

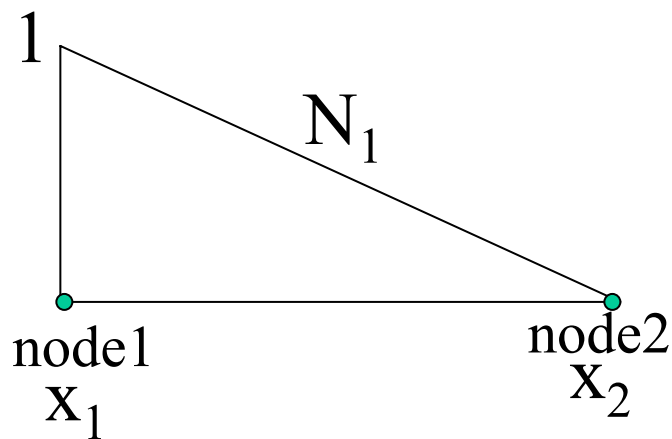
k_e is element stiffness matrix

$$k_e = \frac{AE}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

CONTRIBUTION OF THE FORCES TO PE

BODY FORCES

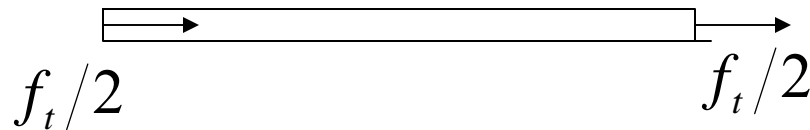
$$\begin{aligned}\int q^T N^T f A dx &= f A q^T \int N^T dx \\ &= f A q^T \begin{bmatrix} \int N_1 dx \\ \int N_2 dx \end{bmatrix}\end{aligned}$$



$$\begin{aligned}\int N_1 dx &= \text{area under the triangle.} \\ &= (x_2 - x_1) / 2 = L_e / 2\end{aligned}$$

$$\begin{aligned}
&= fAq^T \frac{l_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= q^T \frac{fAl_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= q^T \overrightarrow{f_e} = \textit{element body force vector}
\end{aligned}$$

Thus, body force on element gets split equally at two nodes.



TRACTIVE FORCES

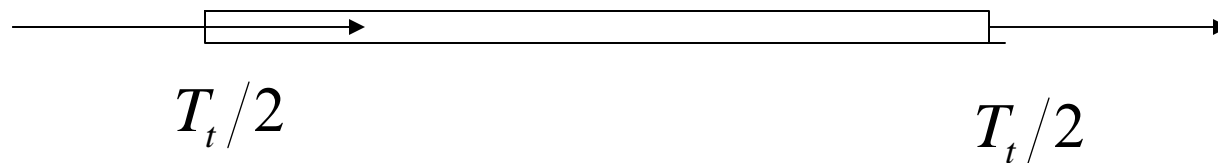
$$\int u^T T dx = \int q^T N^T T dx$$

$$= q^T T \begin{bmatrix} \int N_1 dx \\ \int N_2 dx \end{bmatrix}$$

$$= q^T \frac{T l_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= q^T T_e$$

[TOTAL TRACTIVE FORCE ON THE ELEMENT CAN BE ASSUMED TO BE SPLIT EQUALLY AT THE NODES]



POTENTIAL ENERGY

Potential energy for an element is-

$$\Pi_e = \frac{1}{2} q^T k_e q - q^T f_e - q^T T_e - \sum q^T P$$

Total Potential energy-

$$\Pi = \sum \Pi_e = \sum \frac{1}{2} q^T k_e q - \sum q^T F_e$$

$$F_e = \left[\frac{fAl_e}{2} + \frac{Tl_e}{2} \cdot \frac{fAl_e}{2} + \frac{Tl_e}{2} \right]$$

CONSIDER A SET OF ELEMENTS

Element No.	Node 1	Node 2
1	1	2
2	2	3
3	3	4
4	4	5
5	5	6

1	e_1	2	e_2	3	e_3	4	e_4	5	e_5	6
Q_1		Q_2		Q_3		Q_4		Q_5		Q_6

$$[q_1 \quad q_2]^T [q_2 \quad q_3]^T [q_3 \quad q_4]^T [q_4 \quad q_5]^T$$

ELEMENT DISPLACEMENT MATRIX

$$Q = [q_1 \quad q_2 \quad q_3 \quad - \quad - \quad - \quad q_n]^T$$

GLOBAL DISPLACEMENT MATRIX

$$\frac{E_1 A_1}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{E_2 A_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{E_3 A_3}{l_3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \dots$$

ELEMENT STIFFNESS MATRIX

$$K = \begin{bmatrix} \frac{E_1 A_1}{l_1} & -\frac{E_1 A_1}{l_1} & 0 & 0 & - & - & n \\ \frac{E_1 A_1}{l_1} & \frac{E_1 A_1}{l_1} + \frac{E_2 A_2}{l_2} & -\frac{E_2 A_2}{l_2} & 0 & - & - & - \\ 0 & -\frac{E_2 A_2}{l_2} & \frac{E_2 A_2}{l_2} + \frac{E_3 A_3}{l_3} & -\frac{E_3 A_3}{l_3} & & & \\ 0 & 0 & -\frac{E_3 A_3}{l_3} & \frac{E_3 A_3}{l_3} + \frac{E_4 A_4}{l_4} & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \frac{E_n A_n}{l_n} \end{bmatrix}$$



GLOBAL STIFFNESS MATRIX

1 2 3 4 5 6 n

$$\mathbf{K} = \begin{matrix}
 1 & \left[\begin{array}{cccc|cccc}
 k^1_{11} & k^1_{12} & 0 & 0 & 0 & 0 & \dots & 0 \\
 k^1_{21} & k^1_{22} + k^2_{11} & k^2_{12} & 0 & 0 & 0 & \dots & 0 \\
 0 & k^2_{21} & k^2_{22} + k^3_{11} & k^3_{12} & 0 & 0 & \dots & 0 \\
 0 & 0 & k^3_{21} & k^3_{22} + k^4_{11} & k^4_{12} & 0 & \dots & 0 \\
 5. & & & & & & & \\
 \cdot & & & & & & & \\
 n & \left[\begin{array}{cccc|cc}
 0 & 0 & \dots & & k^m_{21} & k^m_{22} \\
 \end{array} \right]
 \end{matrix}$$

where

$k^i_{11}, k^i_{12}, k^i_{21}, k^i_{22}$: stiffness matrix elements of element number i .

PROPERTIES OF K

- Symmetric
- Banded
- Can be sparse (if numbering is not appropriate)
- Is $N \times N$ (where N is the number of nodes in a 1 D problem)

SPARSE 'K' MATRIX

1	6	2	3	4	5
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Elem No.	Node 1	Node 2
1	1	6
2	6	2
3	2	3
4	3	4
5	4	5

THE RESULTING K MATRIX: SPARSE AND NON-BANDED

$$\mathbf{K} = \begin{array}{c}
 \begin{array}{cccccc}
 1 & & & & & \\
 & 2 & & & & \\
 & & 3 & & & \\
 & & & 4 & & \\
 & & & & 5 & \\
 & & & & & 6
 \end{array} \\
 \left[\begin{array}{cccccc}
 \mathbf{k}_{11}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{12}^1 \\
 \mathbf{0} & \mathbf{k}_{22}^2 + \mathbf{k}_{11}^3 & \mathbf{k}_{12}^3 & \mathbf{0} & \mathbf{0} & \mathbf{k}_{12}^2 \\
 \mathbf{0} & \mathbf{k}_{21}^3 & \mathbf{k}_{11}^4 + \mathbf{k}_{22}^3 & \mathbf{k}_{12}^4 & \mathbf{0} & \mathbf{0} \\
 \mathbf{0} & \mathbf{0} & \mathbf{k}_{21}^4 & \mathbf{k}_{22}^4 + \mathbf{k}_{11}^5 & \mathbf{k}_{12}^5 & \mathbf{0} \\
 \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{21}^5 & \mathbf{k}_{22}^5 & \mathbf{0} \\
 \mathbf{k}_{21}^1 & \mathbf{k}_{12}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{22}^1 + \mathbf{k}_{11}^2
 \end{array} \right]
 \end{array}$$

ASSEMBLING GLOBAL FORCE MATRICES
FROM ELEMENT MATRICES

$$\mathbf{F} = [(\mathbf{F}_{11}^e + \mathbf{P}_1)(\mathbf{F}_{12}^e + \mathbf{F}_{12}^e + \mathbf{P}_2)(\mathbf{F}_{22}^e + \mathbf{F}_{31}^e + \mathbf{P}_3)$$
$$(\mathbf{F}_{32}^e + \mathbf{F}_{41}^e + \mathbf{P}_4) \dots]^T$$

$$= \sum (\mathbf{f}_i^e + \mathbf{T}_i^e) + \mathbf{P} = \sum \mathbf{F}_i^e + \mathbf{P}$$

Where

\mathbf{F}_{i1}^e : force on node 1 of element i

\mathbf{F}_{i2}^e : force on node 2 of element I

\mathbf{P}_i : point load on node number i

$$\left[\begin{array}{cc} \frac{fA l_i}{2} + \frac{Tl_i}{2} & \frac{fA l_i}{2} + \frac{Tl_i}{2} \end{array} \right]$$

ELEMENT FORCE MATRIX

OR

$$[f_{i_1} \quad f_{i_2}]$$

$$F = [f_{11} \quad f_{12} + f_{21} \quad f_{22} + f_{31} \quad f_{32} + f_{41} \quad \cdot \quad \cdot \quad \cdot]^T$$

GLOBAL FORCE MATRIX

TOTAL PE USING GLOBAL MATRICES

$$\begin{aligned}\Pi &= \sum 1/2 * q^T * k_e * q - \sum q^T * f^e - \sum q^T * T^e - \sum P_i * u_i \\ &= \sum 1/2 * q^T * k_e * q - \sum q^T * F^e - \sum P_i * u_i\end{aligned}$$

$$= \boxed{1/2 * Q^T * K * Q - Q^T * F}$$

where

$$Q = [q_1 \ q_2 \ q_3 \ q_4 \ \dots \ q_n]^T$$

K : Global stiffness matrix

F : Global Force matrix

GALERKIN 'S APPROACH

$$\int_v \sigma^T \epsilon(\phi) dV - \int_v \sigma^T f dV - \int_s \phi T dS - \sum \phi^T P = 0$$

$$\phi = \phi(x)$$

$$\Rightarrow \epsilon(\phi) = \frac{d\phi}{dx}$$

$$\phi = N \Psi$$

$$\Rightarrow \epsilon(\phi) = B \Psi$$

$$\Psi = [\phi_1 \quad \phi_2]^T$$

$$u = Nq$$

$$\epsilon = Bq$$

$$\sigma = EBq$$

INTERNAL WORK DONE

$$\begin{aligned}\int_V \sigma^T \epsilon(\phi) dV &= \int q^T B^T E B \Psi A_e dx \\ &= q^T E A_e B^T B \Psi \int dx \\ &= q^T [E A_e l_e B^T B] \Psi \\ &= q^T \left[\frac{E A_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right] \Psi \\ &= q^T k_e \Psi\end{aligned}$$

Virtual work due
To body forces-

$$\begin{aligned}\int_V \phi^T f dV &= \int \Psi^T N^T f A_e dx \\ &= \Psi^T f A_e \int N^T dx \\ &= \Psi^T f \frac{A_e l_e}{2} \int N^T d\xi \\ &= \Psi^T f \frac{A_e l_e}{2} \begin{bmatrix} \int N_1 d\xi \\ \int N_2 d\xi \end{bmatrix} \\ &= \Psi^T f \frac{A_e l_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \Psi^T F_e\end{aligned}$$

SIMILARLY TRACTIVE WORK DONE

$$\int_{\nu} \phi^T T dx = \Psi^T T_e \text{ where } T_e = \frac{Tl_e}{2} [1 \quad 1]^T$$

Thus Galerkin's Equation Becomes

$$\int_{\nu} \sigma^T \in (\phi) dv - \int_S \phi^T f dv - \int_S \phi^T t ds - \Sigma \phi^T P = 0$$

$$\sum q^T k_e \Psi - \sum \Psi^T f_e - \sum \Psi^T T_e - \Psi^T P = 0$$

$$\Rightarrow \sum_e q^T k_e \Psi - \sum_e \Psi^T F_e = 0$$

$$[\Psi_1 \cdots \cdots \Psi_i \Psi_{i+1} \cdots \cdots \Psi_n]$$

$$\left[\begin{array}{cccc} \frac{E_1 A_1}{l_1} & -\frac{E_1 A_1}{l_1} & & \\ -\frac{E_1 A_1}{l_1} & \frac{E_1 A_1}{l_1} + \frac{E_2 A_2}{l_2} & & \\ & & - & \\ & & & - \\ & & & \frac{E_{i-1} A_{i-1}}{l_{i-1}} + \frac{E_i A_i}{l_i} & -\frac{E_i A_i}{l_i} \\ & & & -\frac{E_i A_i}{l_i} & \frac{E_i A_i}{l_i} + \frac{E_{i+1} A_{i+1}}{l_{i+1}} \end{array} \right] Q$$

$$\left[\begin{array}{cccc} \dots & \dots & \Psi_i \frac{E_i A_i}{l_i} & -\Psi_{i+1} \frac{E_i A_i}{l_i} & -\frac{\Psi_i E_i A_i}{l_i} + \frac{\Psi_{i+1} E_i A_i}{l_i} \end{array} \right] \begin{bmatrix} Q_1 \\ \cdot \\ \cdot \\ Q_i \\ Q_{i+1} \\ \cdot \end{bmatrix}$$

In global form-

$$\Psi^T Q K - \Psi^T F = 0 \dots \dots \text{Galerkin's approach}$$

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F \dots \dots \text{P.E. approach}$$

TO SOLVE THE PROBLEM

- **Find Deformations Q_i 's**
 1. **Define Boundary conditions**
 2. **Apply minimization of PE**
- **Find strains $\varepsilon = \mathbf{B} * \mathbf{q}$**
Find stresses $\sigma = \mathbf{E} \varepsilon = \mathbf{E} \mathbf{B} \mathbf{q}$

SOLVING FOR Q

$$Q = [Q_1 \quad Q_2 \quad \dots \quad Q_n]^T$$

$$F = [F_1 \quad F_2 \quad \dots \quad F_n]^T$$

$$K = \begin{bmatrix} K_{11} & K_{12} & \cdot & \cdot & K_{1n} \\ K_{21} & K_{22} & \cdot & \cdot & K_{2n} \\ \cdot & & & & \\ \cdot & & & & \\ K_{n1} & K_{n2} & & & K_{nn} \end{bmatrix}$$

$$\begin{aligned}
\Pi &= \frac{1}{2} \left(\begin{array}{l} Q_1 K_{11} Q_1 + Q_1 K_{12} Q_2 + Q_1 K_{13} Q_3 + \dots + Q_1 K_{1n} Q_n + \\ Q_2 K_{21} Q_1 + Q_2 K_{22} Q_2 + Q_2 K_{23} Q_3 + \dots + Q_2 K_{2n} Q_n + \\ \cdot \\ \cdot \\ Q_n K_{n1} Q_1 + Q_n K_{n2} Q_2 + Q_n K_{n3} Q_3 + \dots + Q_n K_{nn} Q_n \end{array} \right) \\
&- (Q_1 F_1 + Q_2 F_2 + Q_3 F_3 + \dots + Q_n F_n) \\
&= \frac{1}{2} Q^T K Q - Q^T F
\end{aligned}$$

BOUNDARY CONDITIONS

$$Q_{p_1} = a_1, \quad Q_{p_2} = a_2 \quad \dots$$

$$[e.g. \quad Q_2 = 0, \quad Q_5 = 0, \quad \dots]$$

– *SINGLE POINT CONSTRAINT*

LET US CONSIDER AS AN EXAMPLE

$$Q_1 = a_1$$

A BECOMES

$$\begin{aligned} \Pi = & \frac{1}{2} (a_1 K_{11} a_1 + a_1 K_{12} Q_2 + a_1 K_{13} Q_3 + \dots + a_1 K_{1n} Q_n + \\ & + Q_2 K_{21} a_1 + Q_2 K_{22} Q_2 + Q_2 K_{23} Q_3 + \dots + Q_2 K_{2n} Q_n + \\ & + a_3 K_{31} a_1 + Q_3 K_{32} Q_2 + \\ & \cdot \\ & \cdot \\ & Q_n K_{n1} a_1 + Q_n K_{n2} a_2 + \dots + Q_n K_{nn} Q_n) \\ & - (a_1 F_1 + Q_2 F_2 + Q_3 F_3 + \dots + Q_n F_n) \end{aligned}$$

MINIMUM P.E. PRINCIPLE :

$$\frac{\partial \Pi}{\partial Q_i} = 0$$

$$\frac{\partial \Pi}{\partial Q_2} = 0 \left\{ \begin{array}{l} \frac{1}{2} (K_{21} a_1 + Q_2 K_{22} + K_{23} Q_3 + \dots K_{2n} Q_n + \\ K_{12} a_1 + Q_2 K_{22} + K_{32} Q_3 + \dots K_{n2} Q_n) - F_2 = 0 \end{array} \right.$$

$$\frac{\partial \Pi}{\partial Q_3} = 0 \left\{ \begin{array}{l} \frac{1}{2} (K_{31} a_1 + Q_2 K_{32} + K_{33} Q_3 + \dots K_{3n} Q_n + \\ K_{13} a_1 + Q_2 K_{23} + K_{33} Q_3 + \dots K_{3n} Q_n) - F_3 = 0 \end{array} \right.$$

:

$$\frac{\partial \Pi}{\partial Q_i} = 0 \left\{ \begin{array}{l} \frac{1}{2} (K_{i1} a_1 + Q_2 K_{i2} + K_{i3} Q_3 + \dots + K_{in} Q_n + \\ K_{1i} a_1 + Q_2 K_{2i} + K_{3i} Q_3 + \dots K_{ni} Q_n) - F_i = 0 \end{array} \right.$$

$$Q_2 K_{22} + Q_3 K_{23} + \dots + Q_n K_{2n} = F_2 - K_{21} a_1$$

$$Q_2 K_{32} + Q_3 K_{33} + \dots + Q_n K_{3n} = F_3 - K_{31} a_1$$

.

$$Q_2 K_{i2} + Q_3 K_{i3} + \dots + Q_n K_{in} = F_i - K_{i1} a_1$$

$$\begin{bmatrix} K_{22} & K_{23} & K_{24} & \cdot & \cdot & \cdot & K_{2n} \\ K_{32} & K_{33} & K_{34} & \cdot & \cdot & \cdot & K_{3n} \\ \cdot & & & & & & \\ \cdot & & & & & & \\ K_{n2} & K_{n3} & K_{n4} & \cdot & \cdot & \cdot & K_{nn} \end{bmatrix} \begin{bmatrix} Q_2 \\ Q_3 \\ \cdot \\ \cdot \\ Q_n \end{bmatrix} = \begin{bmatrix} F_2 - K_{21} a_1 \\ F_3 - K_{31} a_1 \\ \cdot \\ \cdot \\ F_n - K_{n1} a_1 \end{bmatrix}$$

$$\Rightarrow K^1 Q^1 = F^1$$

K' = OBTAINED FROM K BY DELETING 1ST ROW AND COLUMN

Q' = OBTAINED FROM Q BY DELETING Q₁

F' = OBTAINED FROM F BY DELETING F₁ AND SUBTRACTING K_{i1}a₁ FROM F_i

IF INSTEAD OF $Q_1 = a_1$, WE HAD $Q_i = a_i$ THE SAME STEPS SHALL BE CARRIED OUT BY DOING THESE OPERATIONS ON i^{th} ROW AND COLUMN.

REACTION AT THE SUPPORT (NODE 1 IF $Q_1 = a_1$)

ELIMINATION METHOD

$$K_{11}Q_1 + K_{12}Q_2 + K_{13}Q_3 + \dots + K_{1n}Q_n = F_1 + R_1$$

$$\Rightarrow R_1 = K_{11}Q_1 + K_{12}Q_2 + K_{13}Q_3 + \dots + K_{1n}Q_n - F_1$$

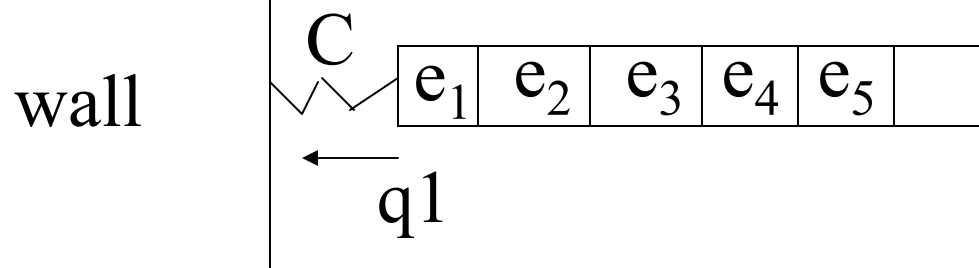
ENALTY APPROACH

ATTACH A SPRING OF STIFFNESS C

DEFLECT THE FIXED END BY a_1

a_1

DEFLECTION OF NODE 1-Q1



AS $C \rightarrow \infty, Q_1 \rightarrow a_1$

-DEFLECTION OF SPRING $\delta = Q_1 - a_1$

$$U_s = \frac{1}{2} C (Q_1 - a_1)^2 [P.E. \text{ of SPRING}]$$

$$\Pi = \frac{1}{2} C (Q_1 - a_1)^2 + \frac{1}{2} Q^T K Q - Q^T F$$

$$= \frac{1}{2} C (Q_1 - a_1)^2 +$$

$$\frac{1}{2} (Q_1 K_{11} Q_1 + Q_1 K_{12} Q_2 + Q_1 K_{13} Q_3 + \dots + Q_1 K_{1n} Q_n +$$

$$Q_2 K_{21} Q_1 + Q_2 K_{22} Q_2 + Q_2 K_{23} Q_3 + \dots + Q_2 K_{2n} Q_n +$$

.

.

$$Q_2 K_{21} Q_1 + Q_2 K_{22} Q_2 + Q_2 K_{23} Q_3 + \dots + Q_2 K_{2n} Q_n)$$

$$- (Q_1 F_1 + Q_2 F_2 + Q_3 F_3 + \dots + Q_n F_n)$$

$$\frac{\partial \Pi}{\partial Q_i} = 0$$

$$\frac{\partial \Pi}{\partial Q_1} = 0 \rightarrow CQ - Ca_1 + K_{11}Q_1 + K_{21}Q_2 + K_{13}Q_3 + \dots K_{1n}Q_n - F_1 = 0$$

$$\frac{\partial \Pi}{\partial Q_2} = 0 \rightarrow K_{21}Q_1 + Q_2 K_{22} + K_{23}Q_3 + \dots K_{2n}Q_n - F_2 = 0$$

$$\frac{\partial \Pi}{\partial Q_3} = 0 \rightarrow K_{31}Q_1 + Q_2 K_{32} + K_{33}Q_3 + \dots K_{3n}Q_n - F_3 = 0$$

$$\frac{\partial \Pi}{\partial Q_n} = 0 \rightarrow K_{n1}Q_1 + Q_2 K_{n2} + K_{n3}Q_3 + \dots K_{nn}Q_n - F_n = 0$$

$$\begin{bmatrix}
 K_{11} + C & K_{12} & K_{13} & \cdot & \cdot & K_{1n} \\
 K_{21} & K_{22} & K_{23} & \cdot & \cdot & K_{2n} \\
 K_{31} & K_{32} & K_{33} & \cdot & \cdot & K_{3n} \\
 \cdot & & & & & \\
 \cdot & & & & & \\
 K_{n1} & K_{n2} & K_{n3} & \cdot & \cdot & K_{nn}
 \end{bmatrix}
 \begin{bmatrix}
 Q_1 \\
 Q_2 \\
 Q_3 \\
 \cdot \\
 \cdot \\
 Q_n
 \end{bmatrix}
 =
 \begin{bmatrix}
 F_1 + Ca_1 \\
 F_2 \\
 F_3 \\
 \cdot \\
 \cdot \\
 F_n
 \end{bmatrix}$$

– ΓΕΙΤΟΝΙΑΚΑ ΜΑΤΡΙΑΚΑ ΕΞΙΣΟΤΗΤΩΝ $(K^{11}, K^{12}, K^{13}, \dots, K^{1n}) \times 10^4$

– ΑΝΙΣΟΤΗΤΕΣ ΕΞΙΣΟΤΗΤΩΝ

– ΧΡΟΝΟΣ ΓΕΝΕΣΗΣ ΚΑΙ ΘΑΝΑΤΟΥ

Multi point constraints-

$$\beta_1 Q_{p1} + \beta_2 Q_{p2} = \beta_0$$

for example – $a_1 = 2a_2$

we can solve it by penalty approach

$$\Pi_m = \frac{1}{2} Q^T K Q + \frac{1}{2} C (\beta_1 Q_{p1} + \beta_2 Q_{p2} - \beta_0) - Q^T F$$

Since C has very large value, P.E. takes minimum value

When $(\beta_1 Q_{p1} + \beta_2 Q_{p2} - \beta_0)$ is minimum.

The modified stiffness and force matrices are-

$$\begin{bmatrix} k_{p1p1} & k_{p1p2} \\ k_{p2p1} & k_{p2p2} \end{bmatrix} \rightarrow \begin{bmatrix} k_{p1p1} + C\beta_1^2 & k_{p1p2} + C\beta_1\beta_2 \\ k_{p2p1} + C\beta_1\beta_2 & k_{p2p2} + C\beta_2^2 \end{bmatrix}$$

$$\begin{Bmatrix} F_{p1} \\ F_{p2} \end{Bmatrix} \rightarrow \begin{Bmatrix} F_{p1} + C\beta_0\beta_1 \\ F_{p2} + C\beta_0\beta_2 \end{Bmatrix}$$

Reactions at support are given by-

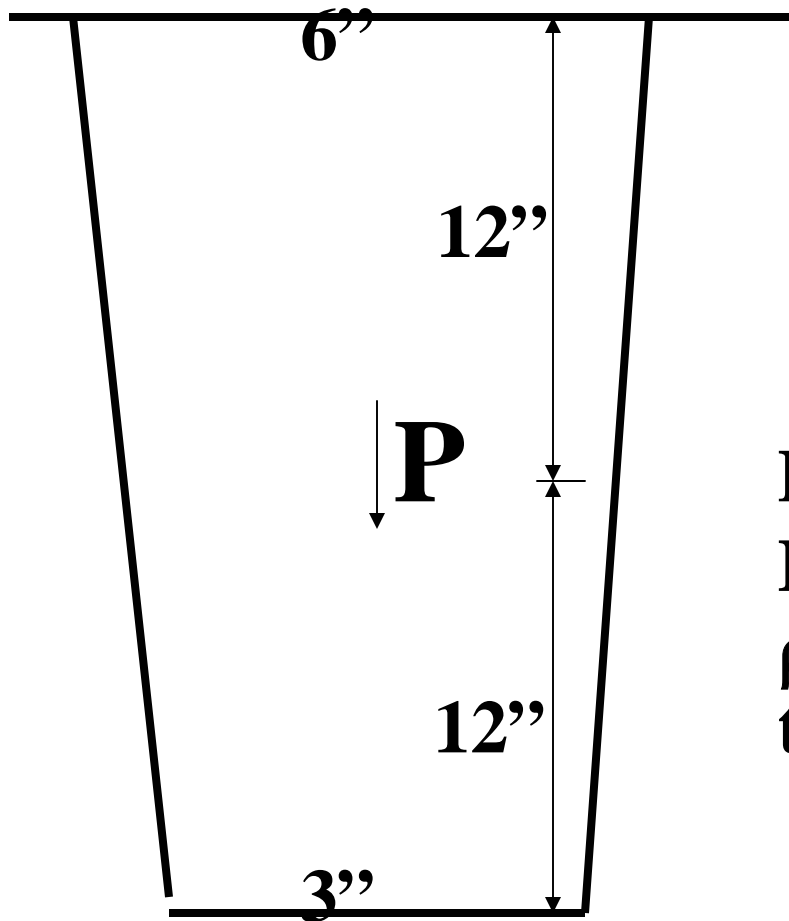
$$R_{p1} = -C\beta_1(\beta_1Q_{p1} + \beta_2Q_{p2} - \beta_0)$$

$$R_{p2} = -C\beta_2(\beta_1Q_{p1} + \beta_2Q_{p2} - \beta_0)$$

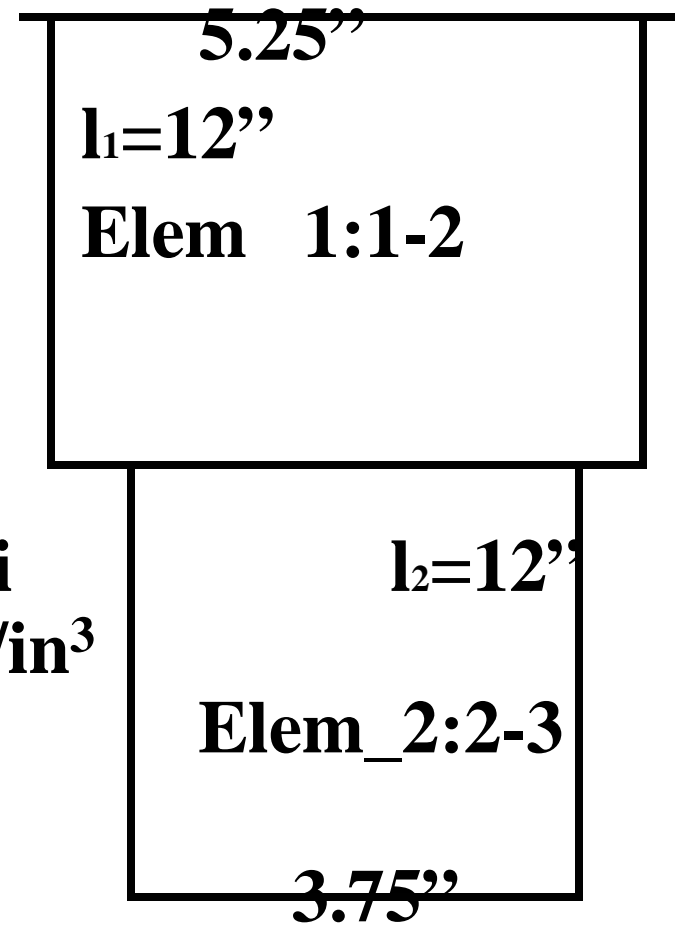
STEPS INVOLVED IN SOLVING A 1D FE PROBLEM

-] ● Make the Geometric Model
- Make an FE Mesh
- Define the Loading
- Develop Element and Global Matrices
- Define Boundary Conditions
- Develop Modified Global Matrices
- Solving Using Numerical Techniques

A SIMPLE PROBLEM



$P=100\text{lb}$
 $E=3 \times 10^7 \text{ psi}$
 $\rho=0.2836\text{lb/in}^3$
 $t=1''$



$$\mathbf{k}^1 = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 15.75 * 10^7 / 12 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{k}^2 = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 11.25 * 10^7 / 12 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{f}^1 = \rho A L / 2 [1 \ 1]^T = 5.25 * 12 * 0.2836 / 2 [1 \ 1]^T$$

$$\mathbf{f}^2 = \rho A L / 2 [1 \ 1]^T = 3.75 * 12 * 0.2836 / 2 [1 \ 1]^T$$

GLOBAL MATRICES

$$\mathbf{K} = 3 \times 10^7 / 12 \begin{bmatrix} 5.25 & -5.25 & 0 \\ -5.25 & 9.00 & -3.75 \\ 0 & -3.75 & 3.75 \end{bmatrix}$$

$$\mathbf{F} = [8.93 \ 15.31 + 100 \ 6.38]^T$$

BOUNDARY CONDITIONS

$$Q_1 = 0$$

MODIFIED MATRICES

$$\mathbf{K}' = 3 \times 10^7 / 12 \begin{bmatrix} 5.25 & -5.25 & 0 \\ -5.25 & 9.00 & -3.75 \\ 0 & -3.75 & 3.75 \end{bmatrix}$$

$$= 3 \times 10^7 / 12 \begin{bmatrix} 9 & -3.75 \\ -3.75 & 3.75 \end{bmatrix}$$

$$\mathbf{F}' = [8.93 \quad (15.31 + 100 - \mathbf{K}_{21} * \mathbf{a}_1) \quad (6.38 - \mathbf{K}_{31} * \mathbf{a}_1)]^T$$

$$= [115.31 \quad 6.38]^T$$

$$\mathbf{Q}' = [\mathbf{Q}_2 \quad \mathbf{Q}_3]^T$$

FINAL EQUATIONS

$$\mathbf{K}'\mathbf{Q}' = \mathbf{F}'$$

Solving we get:

$$\mathbf{Q}' = [0.9272 \ 0.9953] \times 10^{-5}$$

$$\mathbf{Q} = [0 \ 0.9272 \ 0.9953] \times 10^{-5}$$

STRESSES:

$$\begin{aligned}\sigma^1 &= \mathbf{EB}^1\mathbf{q} = 3 \times 10^7 * 1/12 [1 \ -1][0 \ 0.9272 \times 10^{-5}]^T \\ &= 23.18 \text{ psi}\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \mathbf{EB}^2\mathbf{q} = 3 \times 10^7 * 1/12 [1 \ -1][0.9273 \ 0.9953]^T * 10^{-5} \\ &= 1.70 \text{ psi}\end{aligned}$$