Beams and frames

• Beams are slender members used for supporting transverse loading.
• Beams with cross sections symmetric with respect to loading are considered.

\[ \sigma = -\frac{M}{I} y \]
\[ \varepsilon = \sigma / E \]
\[ \frac{d^2v}{dx^2} = \frac{M}{EI} \]
Potential energy approach

Strain energy in an element of length \(dx\) is

\[
dU = \frac{1}{2} \int_{A} \sigma \varepsilon dA dx
\]

\[
= \frac{1}{2} \left( \frac{M^2}{EI^2} \int_{A} y^2 dA \right) dx
\]

\[
\int_{A} y^2 dA \text{ is the moment of inertia } I
\]

The total strain energy for the beam is given by-

\[
U = \frac{1}{2} \int_{0}^{L} EI \left( \frac{d^2 v}{dx^2} \right) dx
\]
Potential energy of the beam is then given by-

$$\Pi = \frac{1}{2} \int_0^L EI \left( \frac{d^2 v}{dx^2} \right) dx - \int_0^L pv dx - \sum_m p_m v_m - \sum_k M_k v'_k$$

Where-

- $p$ is the distributed load per unit length
- $p_m$ is the point load at point $m$.
- $M_k$ is the moment of couple applied at point $k$
- $v_m$ is the deflection at point $m$
- $v'_k$ is the slope at point $k$. 
Galerkin’s Approach

Here we start from equilibrium of an elemental length.

\[ \frac{dV}{dx} = p \]
\[ \frac{dM}{dx} = V \]

\[ \frac{d^2v}{dx^2} = \frac{M}{EI} \]

For approximate solution by Galerkin’s approach-

\[ \int_0^L \left[ \frac{d}{dx^2} \left( EI \frac{d^2v}{dx^2} \right) - p \right] \Phi \, dx = 0 \]

\( \Phi \) is an arbitrary function using same basic functions as \( v \).
Integrating the first term by parts and splitting the interval 0 to L to (0 to $x_m$), ($x_m$ to $x_k$) and ($x_k$ to L) we get-

\[
\int_0^L EI \frac{d^2v}{dx^2} \frac{d^2\Phi}{dx^2} \, dx - \int_0^L p\Phi \, dx + \frac{d}{dx} \left( EI \frac{d^2v}{dx^2} \right) \left. \Phi \right|_0^{x_m} \\
+ \frac{d}{dx} \left( EI \frac{d^2v}{dx^2} \right) \left. \Phi \right|_{x_m}^{x_k} - EI \frac{d^2v}{dx^2} \left. \frac{d\Phi}{dx} \right|_0^{x_k} - EI \frac{d^2v}{dx^2} \left. \frac{d\Phi}{dx} \right|_{x_k}^L = 0
\]

Further simplifying-

\[
\int_0^L EI \frac{d^2v}{dx^2} \frac{d^2\Phi}{dx^2} \, dx - \int_0^L p\Phi \, dx - \sum_m p_m \Phi_m - \sum_k M_k \Phi'_k = 0
\]

$\Phi$ and $M$ are zero at support..at $x_m$ shear force is $p_m$ and at $x_k$ Bending moment is -$M_k$
**FINITE ELEMENT FORMULATION**

- Beam is divided into elements...each node has two degrees of freedom.
- Degree of freedom of node \( j \) are \( Q_{2j-1} \) and \( Q_{2j} \)
- \( Q_{2j-1} \) is transverse displacement and \( Q_{2j} \) is slope or rotation.

\[
\begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_{10}
\end{bmatrix}
= [Q_1, Q_2, Q_3 \ldots Q_{10}]^T
\]

\( Q \) is the global displacement vector.
Local coordinates-

\[ q = [q_1, q_2, q_3, q_4]^T \]

= \[ [v_1, v_1', v_2, v_2'] \]

• The shape functions for interpolating \( v \) on an element are defined in terms of \( \zeta \) from \(-1\) to \(1\).
• The shape functions for beam elements differ from those defined earlier. Therefore, we define ‘Hermite Shape Functions’
Each Hermite shape function is of cubic order represented by-

\[ H_i = a_i + b_i \zeta + c_i \zeta^2 + d_i \zeta^3 \ldots i = 1,2,3,4 \]

The condition given in following table must be satisfied.

<table>
<thead>
<tr>
<th></th>
<th>$H_1$</th>
<th>$H'_1$</th>
<th>$H_2$</th>
<th>$H'_2$</th>
<th>$H_3$</th>
<th>$H'_3$</th>
<th>$H_4$</th>
<th>$H'_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta = -1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\zeta = 1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Finding out values of coefficients and simplifying,

\[ H_1 = \frac{1}{4} (1 - \zeta)^2 (2 + \zeta) \]

\[ H_2 = \frac{1}{4} (1 - \zeta)^2 (\zeta + 1) \]

\[ H_3 = \frac{1}{4} (1 + \zeta)^2 (2 + \zeta) \]

\[ H_4 = \frac{1}{4} (1 + \zeta)^2 (\zeta - 1) \]
Hermite functions can be used to write $v$ in the form-

$$v(\zeta) = H_1 v_1 + H_2 \left( \frac{dv}{d\zeta} \right)_1 + H_3 v_3 + H_4 \left( \frac{dv}{d\zeta} \right)_2$$

The coordinates transform by relationship-

$$x = \frac{1-\zeta}{2} x_1 + \frac{1+\zeta}{2} x_2$$

$$= \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2} \zeta$$

$$dx = \frac{l_e}{2} d\zeta$$

$(x_2-x_1)$ is the length of element $l_e$
\[ \frac{dv}{d\zeta} = \frac{l_e}{2} \frac{dv}{dx} \]

Therefore,

\[ v(\zeta) = H_1 q_1 + \frac{l_e}{2} H_2 q_2 + H_3 q_3 + \frac{l_e}{2} H_4 q_4 \]

\[ v = H q \]

where

\[ H = \begin{bmatrix} H_1, \frac{l_e}{2} H_2, H_3, \frac{l_e}{2} H_4 \end{bmatrix} \]
\[ U_e = \frac{1}{2} \int_{l_e}^{2} EI \left( \frac{d^2v}{dx^2} \right) dx \]

\[ \frac{dv}{dx} = \frac{2}{l_e} \frac{dv}{d\zeta} \quad \text{and} \quad \frac{d^2v}{dx^2} = \frac{4}{l_e} \frac{d^2v}{d\zeta^2} \]

substituting in above equation

\[ \frac{d^2v}{dx^2} = q^T \frac{16}{l_e^4} \left( \frac{d^2H}{d\zeta^2} \right)^T \left( \frac{d^2H}{d\zeta^2} \right) q \]

Where-

\[
\begin{pmatrix}
\frac{d^2H}{d\zeta^2}
\end{pmatrix} = \begin{bmatrix}
\frac{3}{2} \zeta, \frac{-1+3\zeta}{2} \frac{l_e}{2}, \frac{-3}{2} \zeta, \frac{1+3\zeta}{2} \frac{l_e}{2}
\end{bmatrix}
\]
\[ U_e = \frac{1}{2} q^T \frac{8EI}{l_e^3} \int_{-1}^{1} \begin{bmatrix}
9/4 \zeta^2 & 3/8 \zeta (-1+3\zeta) l_e & -9/4 \zeta^2 & 3/8 \zeta (1+3\zeta) l_e \\
\left(\frac{-1+3\zeta}{4}\right)^2 l_e^2 & \frac{-3}{8} \zeta (-1+3\zeta) l_e & \frac{-1+9\zeta^2}{16} l_e^2 \\
\end{bmatrix} \text{symmetric} \begin{bmatrix}
9/4 \zeta^2 \\
-3/8 \zeta (1+3\zeta) l_e \\
\left(\frac{1+3\zeta}{4}\right)^2 l_e^2 \\
\end{bmatrix} d\zeta q \]

Note that-
\[
\int_{-1}^{1} \zeta^2 d\zeta = \frac{2}{3} \quad \int_{-1}^{1} \zeta d\zeta = 0 \quad \int_{-1}^{1} d\zeta = 2
\]

This result can be written as-
\[ U_e = \frac{1}{2} q^T k_e q \]
Where $K_e$ is element stiffness matrix given by

$$
k_e = \frac{EI}{l^3_e} \begin{bmatrix}
12 & 6l_e & -12 & 6l_e \\
6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\
-12 & -6l_e & 12 & -6l_e \\
6l_e & 2l_e^2 & -6l_e & 4l_e^2
\end{bmatrix}
$$

It can be seen that it is a symmetric matrix.
Load vector-

• We assume the uniformly distributed load $p$ over the element.

$$\int_{l_e} p v dx = \left( \frac{p l_e}{2} \int_{-1}^{1} H d \zeta \right) q$$

Substituting the value of $H$ we get-

$$\int_{l_e} p v dx = f^e q$$

where

$$f^e = \begin{bmatrix} \frac{p l_e}{2} & \frac{p l_e^2}{12} & \frac{p l_e}{2} & -\frac{p l_e^2}{12} \end{bmatrix}^T$$
This is equivalent to the element shown below-

The point loads $P_m$ and $M_k$ are readily taken care of by introducing the nodes at the point of application.
Introducing local-global correspondence from potential energy approach we get-

$$\Pi = \frac{1}{2} Q^T KQ - QF$$

And from Galerkin’s approach we get-

$$\Psi^T KQ - \Psi^T F = 0$$

*where $\Psi = \text{admissible virtual displacement vector}$.}
BOUNDARY CONSIDERATIONS

• Let $Q_r = a$ single point BC

• Following Penalty approach, add $1/2C(Q_r-a)^2$ to $\Pi$

• $C$ represents stiffness which is large in comparison with beam stiffness terms.

• $C$ is added to $K_{rr}$ and $Ca$ is added to $F_r$ to get-

  $$KQ = F$$

• These equations are solved to get nodal displacements.

\[
\begin{align*}
\text{Dof} &= (2i-1) \\
\text{Dof} &= 2i
\end{align*}
\]
Shear Force and Bending Moment -

We have,

\[ M = EI \frac{d^2v}{dx^2} \quad V = \frac{dM}{dx} \quad \text{and} \quad v = Hq \]

\[
\begin{bmatrix}
R_1 \\
R_2 \\
R_3 \\
R_4
\end{bmatrix} = \frac{EI}{l_e^3} \begin{bmatrix}
12 & 6l_e & -12 & 6l_e \\
6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\
-12 & -6l_e & 12 & -6l_e \\
6l_e & 2l_e^2 & -6l_e & 4l_e^2
\end{bmatrix} \begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix} + \begin{bmatrix}
-p l_e/2 \\
-p l_e^2/12 \\
-p l_e/2 \\
p l_e^2/12
\end{bmatrix}
\]

\[ V_1 = R_1 \quad V_2 = -R_3 \quad M_1 = -R_2 \quad M_2 = R_4 \]
Beams on elastic support

• Shafts supported on ball, roller, journal bearings
• Large beams supported on elastic walls.
• Beam supported on soil (Winkler foundation).

• Stiffness of support contributes towards PE.
• Let ‘s’ be the stiffness of support per unit length.

additional term = \frac{1}{2} \int_{0}^{l} s v^2 \, dx

v = Hq

= \frac{1}{2} \sum_{e} q^T s \int_{e} H^T H \, dx \, q
\[
\frac{1}{2} \sum_e q^T k^s_e
\]

where \( k^s_e \) is stiffness matrix for elastic foundation

\[
k^s_e = \frac{sl_e}{420} \begin{bmatrix}
156 & 22l_e & 54 & -13l_e \\
22l_e & 4l^2_e & 13l_e & -3l^2_e \\
54 & 13l_e & 156 & 22l_e \\
-13l_e & -3l^2_e & -22l_e & 4l^2_e
\end{bmatrix}
\]
PLANE FRAMES

• Plane structure with rigidly connected members.

• Similar to beams except that axial loads and deformations are present.

• We have 2 displacements and 1 rotation at each node.
• 3 dof at each node.

\[ q = [q_1, q_2, q_3, q_4, q_5, q_6]^T \]
Global and local coordinate systems
\[ q' = [q'_1, q'_2, q'_3, q'_4, q'_5, q'_6] \]

\[ l, m \] are the direction cosines of local coordinate system. \( X'Y' \)
- \( l = \cos(\theta) \)
- \( m = \sin(\theta) \)

We can see from the figure that-
- \( q_3 = q'_3 \)
- \( q_3 = q'_6 \)
which are rotations with respect to body.
\[ q' = Lq \]

Where-

\[
L = \begin{bmatrix}
  l & m & 0 & 0 & 0 & 0 \\
  -m & l & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & l & m & 0 \\
  0 & 0 & 0 & -m & l & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\( q'_{2}, q'_{3}, q'_{5} \) and \( q'_{6} \) are beam element dof while \( q'_{1} \) and \( q'_{4} \) are like rod element dof.
Combining two stiffness and rearranging at proper locations we get element stiffness matrix as-

\[ k'^e = \begin{bmatrix}
\frac{EA}{l_e} & 0 & 0 & -\frac{EA}{l_e} & 0 & 0 \\
0 & \frac{12EI}{l_e^3} & \frac{6EI}{l_e^2} & 0 & -\frac{12EI}{l_e^3} & \frac{6EI}{l_e^2} \\
0 & \frac{6EI}{l_e^2} & \frac{4EI}{l_e} & 0 & -\frac{6EI}{l_e^2} & \frac{2EI}{l_e} \\
-\frac{EA}{l_e} & 0 & 0 & \frac{EA}{l_e} & 0 & 0 \\
0 & -\frac{12EI}{l_e^3} & -\frac{6EI}{l_e^2} & 0 & \frac{12EI}{l_e^3} & -\frac{6EI}{l_e^2} \\
0 & \frac{6EI}{l_e^2} & \frac{2EI}{l_e} & 0 & -\frac{6EI}{l_e^2} & \frac{4EI}{l_e}
\end{bmatrix} \]
Strain energy of an element is given by-

\[ U_e = \frac{1}{2} q^T k^e q' \]

\[ = \frac{1}{2} q^T L^T k^e L q \]

by Galerkin 's approach,

\[ W_e = \Psi^T k^e q' \]

\[ = \Psi^T L^T k^e L q \]

Element stiffness matrix in global form can be written as-

\[ K^e = L^T k^e L \]
If there is distributed load on member, we have-

\[ q'^T f' = q^T L^T f' \]

where

\[ f' = \begin{bmatrix} 0, \frac{pl_e}{2}, \frac{pl_e^2}{12}, 0, \frac{pl_e}{2}, -\frac{pl_e^2}{12} \end{bmatrix}^T \]

\[ f = L^T f' \]

In global form,

\[ KQ = F \]