POTENTIAL ENERGY, $\Pi$.

The total potential energy of an elastic body, is defined as the sum of total strain energy ($U$) and the work potential ($WP$).

\[ \Pi = U + WP \]
For linear elastic materials, the strain energy per unit volume in the body is “$\frac{1}{2}\sigma^T\varepsilon$”.

For elastic body total strain energy ($U$) is $U = \frac{1}{2} \int \sigma^T \varepsilon dv$. 
The work potential is given by

\[ WP = -\int_V \dot{u}^T f dV - \int_S \dot{u}^T T ds - \sum_i u_i^T P_i \]

The total potential energy for the general elastic body is

\[ \pi = \frac{1}{2} \int \sigma^T \varepsilon dv - \int_V \dot{u}^T f dV - \int_S \dot{u}^T T ds - \sum_i u_i^T P_i \]
Principal of minimum potential energy

For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.
Example

Figure-1
Figure 1 shows a system of springs.

The total potential energy is given by

$$\pi = \frac{1}{2} k_1 \delta_1^2 + \frac{1}{2} k_2 \delta_2^2 + \frac{1}{2} k_3 \delta_3^2 + \frac{1}{2} k_4 \delta_4^2 - F_1 q_1 - F_3 q_3$$

where $\delta_1, \delta_2, \delta_3,$ and $\delta_4$ are extensions of four springs.

Since

$$\delta_1 = q_1 - q_2$$

$$\delta_2 = q_2$$

$$\delta_3 = q_3 - q_2$$

$$\delta_4 = -q_3$$
we have

\[ \pi = \frac{1}{2} k_1 (q_1 - q_2)^2 + \frac{1}{2} k_2 q_2^2 + \frac{1}{2} k_3 (q_3 - q_2)^2 + \frac{1}{2} k_4 q_3^2 - F_1 q_1 - F_3 q_3 \]

where

\( q_1, q_2, \) and \( q_3 \) are the displacements of nodes \( 1, 2 \) and \( 3 \) respectively.
Figure E1.1b
For equilibrium of this 3-DOF system, we need to minimize to $\prod$ with respect to $q_1$, $q_2$, and $q_3$ the three equations are given by

$$\frac{\partial \pi}{\partial q_i} = 0 \quad i = 1, 2, 3$$

which are

$$\frac{\partial \pi}{\partial q_1} = k_1 (q_1 - q_2) - F_1 = 0$$
\[ \frac{\partial \pi}{\partial q_2} = -k_1 (q_1 - q_2) + k_2 q_2 - k_3 (q_3 - q_2) = 0 \]

\[ \frac{\partial \pi}{\partial q_3} = k_3 (q_3 - q_2) + k_4 q_3 - F_3 = 0 \]

Equilibrium equation can be put in the form of

\[ K q = F \] as follows

\[
\begin{bmatrix}
K_1 & -K_1 & 0 \\
-K_1 & K_1 + K_2 + K_3 & -K_3 \\
0 & -K_3 & K_3 + K_4
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3
\end{bmatrix}
= 
\begin{bmatrix}
F_1 \\
0 \\
F_3
\end{bmatrix}
\]
If on the other hand, we proceed to write the equilibrium of the system by considering the equilibrium of each separate node as shown in figure 2. We can write

\[ K_1 \delta_1 = F_1 \]

\[ K_2 \delta_2 - K_1 \delta_1 - K_3 \delta_3 = 0 \]

\[ K_3 \delta_3 - K_4 \delta_4 = F_3 \]

Which is precisely the set of equations represented in Eq-1.
We see clearly that the set of equation “1” is obtained in a routine manner using the potential energy approach, without any reference to the free body diagrams.

This make the potential energy approach attractive for large and complex problems.
RAYLEIGH-RITZ METHOD

Rayleigh-Ritz method involves the construction of an assumed displacement field, say

\[ u = \sum a_i \Phi_i( x, y, z) \quad i = 1 \text{ to } L \]
\[ v = \sum a_j \Phi_j( x, y, z) \quad j = L + 1 \text{ to } M \]
\[ w = \sum a_k \Phi_k( x, y, z) \quad k = M + 1 \text{ to } N \]

\[ N > M > L \]  

**Eq-1**
The functions $\Phi_i$ are usually taken as polynomials. Displacements $u$, $v$, $w$ must satisfy boundary conditions.

Introducing stress-strain and strain-displacement relation Substituting equation – 1 in to $\prod (PE)$
\[ \Pi = \Pi (a_1, a_2, a_3, \ldots, a_r) \]

where \( r = \text{no of independent unknowns} \)

the extremum with respect to \( a_i, (i = 1 \text{ to } r) \)
yields the set of \( r \) equation

\[ \frac{\partial \pi}{\partial a_i} = 0 \quad i = 1, 2, 3, \ldots, r \]
Example

The potential energy of the linear 1-D rod with body force is neglected, is

\[ \pi = \frac{1}{2} \int_{0}^{l} \left[ EA \left( \frac{du}{dx} \right)^2 \right] dx - 2u_1 \]

where \( u_1 = u \quad (x = 1) \)
E = 1, A = 1
let us consider a polynomial function

\[ u = a_1 + a_2 x + a_3 x^3 \]

this must satisfy

\[ u = 0, \text{ at } x = 0 \]

\[ u = 0 \text{ at } x = 2 \]

thus

\[ 0 = a_1 \]

\[ 0 = a_1 + 2a_2 + 4a_3 \]
Figure- 2
Hence
\[ a_2 = -2a_3 \]
\[ u = a_3 (-2x + x^2) \]
\[ u_1 = -a_3 \]

then, \[ \frac{du}{dx} = 2a_3 (-1 + x) \] and

\[ \pi = \frac{1}{2} \int_{0}^{l} \left[ EA \left( \frac{du}{dx} \right)^2 dx - 2u_1 \right] \]

\[ = 2a_3^2 \left( \frac{2}{3} \right) + 2a_3 \]
we set \( \frac{\partial \pi}{\partial a_3} = 0 \)

Resulting in \( a_3 = -0.75 \)

\( u_1 = -a_3 = 0.75 \)

the stress in bar given by

\[
\sigma = E \frac{du}{dx} = 1.5 (1 - x)
\]

exact solution is obtained if piecewise polynomial interpolation is used in the construction of \( u \).
GALERKIN’S METHOD

• Galerkin’s method uses the set of governing equations in the development of an integral form.

• It is usually presented as one of the weighted residual methods.

• Let us consider a general representation of a governing equation on a region “V”

\[ Lu = P \]

Where, “L” as operator operating on “u”
For the one-dimensional rid considered in previous example

Governing equation

\[
\frac{d}{dx} \left( EA \frac{du}{dx} \right) = 0
\]

We may consider L as operator, operating on “u”

\[
\frac{d}{dx} \left( EA \frac{d(\quad)}{dx} \right)
\]
• The exact solution needs to satisfy “Lu=P” at every point x.
• If we seek an approximate solution \( \vec{u} \), if introduces an error \( \varepsilon(x) \), called the residual
  \[
  \varepsilon(x) = L\vec{u} - P
  \]
• The approximate methods revolve around setting the residual relative to a weighting function \( W_i, i = 0 \) to \( n \)
  \[
  \int W_i (L\vec{u} - P) \, dV = 0
  \]
The weighting function $W_i$ are chosen from the basis functions used for constructing

$$\bar{u} = \sum_{i=1}^{n} Q_i G_i$$

- Here, we choose the weighting function to be linear combination of the basis function $G_i$. Specifically, consider an arbitrary function $\phi$. 
Given by

$$\phi = \sum_{i=1}^{n} \phi_i G_i$$

Where the coefficient $\phi_i$ are arbitrary, except for requiring that $\phi$ satisfy boundary conditions were $\mathbf{u}$ is prescribed.
For elastic materials

\[
\int_V \left[ \left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x \right) \Phi_x + \ldots \right] dV = 0
\]

\[
\int_V \frac{\partial \alpha}{\partial x} \vartheta dV = -\int_V \alpha \frac{\partial \theta}{\partial x} dV + \int_S n_x \alpha \theta dS
\]

\[
\int_V \sigma^T \varepsilon(\phi) dV - \int_V \phi^T f dV - \int_S \phi^T T dS - \sum_i \phi^T P
\]
Example

let us consider the problem of the previous example and solve it by Galerkin’s approach. The equilibrium equation is

\[ \frac{d}{dx} EA \frac{du}{dx} = 0 \]

- u=0 at  x=0
- u=0 at  x=0

Multiplying this differential equation by Integrating by parts, we get
Figure- 2
Where \( \phi \) is zero at \( x = 0 \) and \( x = 2 \).

\[
\int_{0}^{2} -EA \frac{du}{dx} \frac{d\phi}{dx} + (\phi EA \frac{du}{dx})_0^1 + (\phi EA \frac{du}{dx})_1^2 = 0
\]

\( EA \frac{du}{dx} \) is the tension in the rod, which takes a jump of magnitude 2 at \( x = 1 \), thus

\[
\int_{0}^{2} -EA \frac{du}{dx} \frac{d\phi}{dx} + 2\phi_1 = 0
\]
Now we use the same polynomial (basis) for $u$ and $\phi$

if $u_1$ and $\phi$ are the value at $x = 1$, thus

$$u = \left(2x - x^2\right)u_1$$

$$\phi = \left(2x - x^2\right)\phi_1$$

• Substituting these and $E = 1$, $A = 1$ in the previous integral yields

$$\phi \left[ -u_1 \int_{0}^{2} (2 - 2x)^2 \, dx + 2 \right] = 0$$
\[ \phi_1 \left( -\frac{8}{3} u_1 + 2 \right) = 0 \]

This is to be satisfied for every \( \phi_1 \). We get

\[ u_1 = 0.75 \]