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Formulae for the Frobenius number in three variables

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ABSTRACT

Text. For positive integers a, b, c that are coprime, the Frobenius number of a, b, c, denoted by $\mathtt{g}(a,b,c)$, is the largest integer that is not expressible by the form $ax+by+cz$ with x, y, z nonnegative integers. We give *exact* formulae for $\mathtt{g}(a,b,c)$ that covers *all* cases of a, b, c.

Video. For a video summary of this paper, please visit https://youtu.be/dv0GSy2MGzw.

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1. Introduction

The `Frobenius Problem` (`FP`) is to determine the largest positive integer that is not representable as a nonnegative integer combination of given positive integers that are coprime. Due to an obvious connection with supplying change in terms of coins of certain fixed denominations, the Frobenius problem is also known as the `Coin Exchange Problem` or as the `Money Changing Problem`. More formally, given positive integers

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$a_1, \ldots, a_n$, with $\gcd(a_1, \ldots, a_n) = 1$, it is well known and not hard to show that for all sufficiently large N the equation

$$a_1 x_1 + \cdots + a_n x_n = N \tag{1}$$

has a solution with nonnegative integers $x_1, \ldots, x_n$. The **Frobenius number** $\mathtt{g}(a_1, \ldots, a_n)$ is the *largest* integer N such that (1) has no solution in nonnegative integers. Although the origin of the problem is attributed to Sylvester [26], who showed that $\mathtt{g}(a_1, a_2) = a_1 a_2 - a_1 - a_2$, an apparent reason for associating the name of Frobenius with this problem is possibly due to the fact that he was largely instrumental in popularizing this problem in his lectures. The Frobenius problem has a rich and long history, with several applications and extensions, and connections to several areas of research. A comprehensive survey covering all aspects of the problem can be found in [18]; also see [8].

Exact determination of the Frobenius number is a difficult problem in general. Brauer [3] found the Frobenius number for consecutive integers, Roberts [20] extended this result to numbers in arithmetic progression (see also [1,29,34]), and Selmer [24] further generalized this to the determination of $\mathtt{g}(a, ha + d, ha + 2d, \ldots, ha + nd)$ (see also [31]). There are only a few other cases where the Frobenius number has been exactly determined for any n variables; refer to [18] for other instances. In the absence of exact results, research on the Frobenius problem has often been focused on sharpening bounds on the Frobenius number and on algorithmic aspects. Although running time of these algorithms is superpolynomial, Kannan [15] gave a method that solved the Frobenius problem in polynomial time for *fixed* number of variables using the concept of covering radius, and Ramírez Alfonsín [17] showed that the problem is NP-hard under Turing reduction.

The purpose of this article is to give exact results for the Frobenius number $\mathtt{g}(a_1, a_2, a_3)$ in all cases. Most of the results in this article appeared in the author's thesis [28], but were not communicated earlier. Although the Frobenius number $\mathtt{g}(a_1, a_2)$ is easy to determine, exact formulae for $\mathtt{g}(a_1, a_2, a_3)$ for all choices of the variables were not previously known and results concerning this were limited to algorithms, bounds and exact results in some special cases.

1.1. A brief overview

We divide our article into three sections. We begin with a brief introduction to the FP in Section 1. In Section 2, we give a historical perspective to the special case of the FP in three variables, and cite two crucial results (Theorems 1 and 2) we use to obtain our formulae. Section 3 contains the formulae for $\mathtt{g}(a, b, c)$. For convenience, we have subdivided this into six subsections. We give two independent sets of formulae, each of which covers all cases of a, b, c. Both sets include the results given in Lemmas 1 and 2; additionally, one set of results is given by Theorems 3 and 5, while the other set is given by results in Theorems 4 and 6. The subcases covered by Theorems 3 and 4 give neat results. Formulae given in Theorems 5 (a), when either of the cases apply, or in Theorem 6 (a), again when either of the cases apply, also give neat results. Having two sets of results means that almost all triples (a, b, c) are covered by either Theorems 3 and 5 (a) or by Theorems 4 and 6 (a).

As a consequence of the well known result of Johnson (Theorem 2), it is no loss of generality to assume that a, b, c are pairwise coprime; we also assume that $a < b < c$. We use another well known theorem, of Brauer & Shockley (Theorem 1), to compute $\mathtt{g}(a, b, c)$. Their result says that $\mathtt{g}(a, b, c)$ is given by $\left(\max_{1\leq i\leq a-1} m_i\right) - a$, where m_i denotes the least positive integer of the form $bx + cy$ $(x, y \geq 0)$ that is congruent to i modulo a. To determine m_i, we introduce two key numbers each of which depend closely on c/b. More specifically, k equals $\lfloor cb^{-1} \rfloor$ and ℓ is congruent to cb^{-1} modulo a. Hence beginning with $bx+cy$, adding 1 to y while either subtracting ℓ from x or adding $a-\ell$ to x results in a number $bx'+cy'$ that is in the same congruence class modulo a. Depending on the relative size of k and ℓ, this increases or decreases the current value of $bx + cy$. Subtracting ℓ from x to offset adding 1 to y leads to our first approach, while the second approach requires simultaneous addition of $a - \ell$ to x and 1 to y. We give arguments to cover all cases with the first approach, but give only a sketch of the proof for the second approach, since it requires an analogous argument. There are two reasons for providing both approaches. First, special cases sometimes easily follow from only one of the two. For instance, the result of Corollary 1 can be deduced from Theorem 3 but only when $c > \frac{1}{2}(a-2)b$ from Theorem 4, and the result of Corollary 2 can be deduced from Theorem 4 but only when $c > 2b$ from Theorem 3. Second, there are many instances of triples (a, b, c) for which both $\Lambda = \Delta$ and $\Lambda' = \Delta'$ hold and for which at least one of $\overline{\Lambda} > \overline{\Delta}$, $\overline{\Delta}' > \overline{\Lambda}'$ hold, making Theorem 6 much more the viable option than Theorem 5.

Lemma 1 is easy to see, and has been stated for the sake of completeness. Lemma 2 covers the case where $\ell \leq k$, where the definitions imply that c must itself be of the form $ax+by$, with $x, y \geq 0$. The nontrivial case is therefore the case where $\ell > k$. It is relatively straightforward to arrive at the formula for $\mathtt{g}(a, b, c)$ when $br < cq$ (Theorem 3) or when $b(\ell - \overline{r}) < c(\overline{q} + 1)$ (Theorem 4), leading to a pair of parallel results. Corollaries 1, 2 and 3 are exact results for $\mathtt{g}(a, b, c)$ that apply to a family of triples and have previously appeared in the literature of the FP; these are deduced as special cases from these two theorems. The other subcase is far more complicated, and necessitates the use and study of a special set $\mathcal{X}$ which we describe in Lemma 11, with analogous definitions and results for the set $\overline{\mathcal{X}}$ in Lemma 12. Formulae for $\mathtt{g}(a, b, c)$ when $br > cq$ are given by Theorem 5, and when $b(\ell-\overline{r}) > c(\overline{q}+1)$ by Theorem 6. There are instances where the formula is not as explicit as one may have hoped for, for instance in Theorem 5 (b) and in Theorem 6 (b). It is for this reason that a parallel attack has been formulated since the same triple may satisfy the requirements in the parallel case, leading to a more easily computable formula. For instance, $\mathtt{g}(100, 101, 139)$ uses the more cumbersome Theorem 5 (b) (Example 5) but falls into Theorem 6 (a) (Example 6).

2. Preliminaries

In this section, we discuss the Frobenius Problem specifically in the case of three variables. There are several algorithms for computing $\mathsf{g}(a_1, a_2, a_3)$, none of which lead to an exact formula. Selmer & Beyer [25] developed an algorithm to compute $\mathsf{g}(a_1, a_2, a_3)$ that required using the continued fraction expansion of a_3/a_2. This was simplified first by Rødseth [21], and later by Davison [6]. Tinaglia [27] proposed a procedure that reduced the computation of $\mathsf{g}(a_1, a_2, a_3)$ to $\mathsf{g}(a_1, r, s)$ where $r \equiv a_2 \bmod a_1$ and $s \equiv a_3 \bmod a_1$. In addition, several of the algorithms to compute the Frobenius number in the general case are computationally comparable to the ones specific to the three variable case, notably those by Böcker & Lipták [2], Greenberg [9], Heap & Lynn [10–12], Nijenhuis [16], Scarf & Shallcross [23], and Wilf [33].

The search for an exact formula for $\mathsf{g}(a_1, a_2, a_3)$ had proved elusive so far. In fact, Curtis [5] showed that the Frobenius number cannot be represented by closed formulae of a certain type. An explicit general formula for computing $\mathsf{g}(a_1, a_2, a_3)$ in terms of the least representable multiples of the three variables was given by Denham [7], Ramírez Alfonsín [19], and Tripathi & Vijay [32].

Theorem 1. (Brauer & Shockley, [4]) *Let $a_1, \ldots, a_n$ be positive integers with* $\gcd(a_1, \ldots, a_n) = 1$. *Let $\Gamma = \Gamma(a_1, \ldots, a_n)$ denote the set of integers of the form $a_1x_1 + \cdots + a_nx_n$ with each $x_i \geq 0$. Then*

$$\mathsf{g}(a_1, \ldots, a_n) = \left(\max_{1 \leq i \leq a_1 - 1} m_i \right) - a_1,$$

where $m_i = \min\big(\Gamma \cap (i)\big)$ and (i) is the residue class of i modulo a_1.

Theorem 2. (Johnson, [13]). *Let $a_1, \ldots, a_n$ be positive integers with* $\gcd(a_1, \ldots, a_n) = 1$. *If* $\gcd(a_2, \ldots, a_n) = d$ *and $a_i = da_i'$ for $i = 2, 3, \ldots, n$, then*

$$\mathsf{g}(a_1, \ldots, a_n) = d \cdot \mathsf{g}(a_1, a_2', \ldots, a_n') + a_1(d-1).$$

3. Formulae for $\mathsf{g}(a, b, c)$

3.1. Key definitions

For positive integers $a_1, \ldots, a_n$ with $\gcd(a_1, \ldots, a_n) = 1$, we write

$$\Gamma(a_1, \ldots, a_n) := \{a_1x_1 + \cdots + a_nx_n : x_i \in \mathbb{N} \cup \{0\}\},$$

and let $\Gamma^c(a_1, \ldots, a_n) = \mathbb{N} \setminus \Gamma(a_1, \ldots, a_n)$. Then the Frobenius number

$$\mathsf{g}(a_1, \ldots, a_n) := \max \Gamma^c(a_1, \ldots, a_n).$$

We deal with the case $n = 3$, and write a_1, a_2, a_3 as a, b, c with $a < b < c$. In view of Theorem 2, it is no loss of generality to assume that a, b, c are *pairwise* coprime.

We give exact results for $\mathsf{g}(a, b, c)$ in terms of two variables k and ℓ, the first of which is the integral part of cb^{-1} and the second the equivalence class of cb^{-1} modulo a. Note that the assumption of pairwise coprimality allows for the second definition.

$$k := \left\lfloor \tfrac{c}{b} \right\rfloor, \quad \ell :\equiv cb^{-1} \pmod{a}.$$

We show that $c \in \Gamma(a, b)$, and consequently, $\mathsf{g}(a, b, c) = \mathsf{g}(a, b)$ if and only if $k \geq \ell$ in Lemma 2. For the most part then, we assume that $\ell > k$. An integral part of our formulae involves the quotient and remainder obtained by dividing a by $a - \ell$. By a parallel argument, we obtain results involving the quotient and remainder obtained by dividing a by ℓ. We know by Theorem 1 that $\mathsf{g}(a, b, c)$ is of the form $bx + cy - a$ with $x, y \geq 0$, and so we seek a pair of nonnegative integers (x_0, y_0) for which $\mathsf{g}(a, b, c) = bx_0 + cy_0 - a$. We also know by the same theorem that $\mathsf{g}(a, b, c)$ is the maximum among the largest integer in $\Gamma^c(b, c) \cap (i)$, taken over all nonzero residue classes (i) modulo a. It is easy to see that $bx + cy$ and $b\big(x + (a - \ell)\big) + c(y + 1)$ are in the same equivalence class modulo a. By repeated applications of this and by breaking up the results into several cases, we determine the pair (x_i, y_i) that corresponds to the *smallest* integer $bx_i + cy_i \in \Gamma(b, c) \cap (i)$ for each nonzero residue class (i) modulo a.

3.2. Preliminary results

We denote the equivalence class containing x modulo a by (x) and the least positive integer in $\Gamma(b, c) \cap (x)$ by $\mathbf{m}(x)$. We begin with the following result.

Lemma 1. *If* $\gcd(a, b) = 1$ *and* $a < b$*, then*

$$\mathsf{g}(a, b, c) = \begin{cases} \mathsf{g}(a, b) & \text{if } c > \mathsf{g}(a, b); \\ \mathsf{g}(a, b) - a & \text{if } c = \mathsf{g}(a, b). \end{cases}$$

Proof. If $c > \mathsf{g}(a, b)$, then $c \in \Gamma(a, b)$, so that $\Gamma(a, b, c) = \Gamma(a, b)$. Therefore $\Gamma^c(a, b, c) = \Gamma^c(a, b)$, and so $\mathsf{g}(a, b, c) = \mathsf{g}(a, b)$.

If $c = \mathsf{g}(a, b)$ and $n < \mathsf{g}(a, b)$, then $n \in \Gamma(a, b, c)$ if and only if $n \in \Gamma(a, b)$. So if $\mathbf{m}^\star(x)$ denotes the least positive integer in $\Gamma(a, b, c) \cap (x)$, we have $\mathbf{m}^\star(x) = \mathbf{m}(x)$ except for $(x) = (c)$. Since $\mathbf{m}^\star(c) = c = \mathsf{g}(a, b)$, and the second part now follows from Theorem 1. $\square$

Henceforth, we restrict our attention to $c < \mathsf{g}(a, b) = ab - a - b$, so that $k \leq a - 2$. In fact, in view of the following result, we may further restrict ourselves to $\ell > k$.

Lemma 2. *If* $\gcd(a, b) = 1$*,* $a < b < c$ *and* $\ell \leq k$*, then* $\mathsf{g}(a, b, c) = \mathsf{g}(a, b)$.

Proof. Since $c \equiv b\ell \pmod{a}$, we can write $c = am + b\ell$ for some $m \in \mathbb{Z}$. But then $m = \frac{1}{a}(c - b\ell) \geq \frac{1}{a}(k - \ell)b \geq 0$. Hence $c \in \Gamma(a, b)$, and so $\mathsf{g}(a, b, c) = \mathsf{g}(a, b)$. □

3.3. A key algorithm, more key definitions and preliminary results

As noted earlier, $\mathsf{g}(a, b, c)$ is of the form $bx + cy - a$ with $x, y \geq 0$. We seek a pair (x_0, y_0) that achieves this, and for brevity, use the notation $\mathbf{v}(x, y) := bx + cy$ and call this the `v-value` of (x, y). By Theorem 1,

$$\mathsf{g}(a, b, c) = \left(\max_{1 \leq i \leq a-1} \mathbf{m}(i) \right) - a = \left(\max_{1 \leq x \leq a-1} \mathbf{m}(bx) \right) - a$$

since $\gcd(a, b) = 1$. For a fixed x_0, $1 \leq x_0 \leq a - 1$, we note that

$$bx + cy \equiv bx_0 \Leftrightarrow b(x - x_0) \equiv -cy \equiv -b\ell y \pmod{a} \Leftrightarrow x \equiv x_0 - \ell y \pmod{a}.$$

Hence the integers $b\big((x_0 - \ell t) \bmod a\big) + ct$, $0 \leq t \leq a - 1$ all belong to the class (bx_0), and we record this as the following result.

Lemma 3. *Let a, b, c be positive integers that are pairwise coprime. Then*

$$\mathsf{g}(a, b, c) = \max_{1 \leq x \leq a-1} \left\{ \min_{0 \leq t \leq a-1} \mathbf{v}\big((x + (a - \ell)t) \bmod a, t\big) \right\} - a.$$

Definition 1. Let $1 \leq x_0 \leq a - 1$. For $1 \leq y_0 \leq a - 2$, the integer $\mathbf{v}(x_0, y_0)$ is said to be a `local minimum` if

$$\mathbf{v}(x_0, y_0) \leq \min\big\{\mathbf{v}\big((x_0 - \ell) \bmod a, y_0 + 1\big), \mathbf{v}\big((x_0 + \ell) \bmod a, y_0 - 1\big)\big\}.$$

If $y_0 = 0$, $\mathbf{v}(x_0, 0)$ is said to be a `local minimum` if $\mathbf{v}(x_0, 0) \leq \mathbf{v}\big((x_0 - \ell) \bmod a, 1\big)$. If $y_0 = a - 1$ $\mathbf{v}(x_0, a - 1)$ is said to be a `local minimum` if $\mathbf{v}(x_0, a - 1) \leq \mathbf{v}\big((x_0 + \ell) \bmod a, a - 2\big)$. We say that two local minima, $\mathbf{v}(x_0, y_0)$ and $\mathbf{v}(x_0', y_0')$, are `consecutive` provided there is no local minimum $\mathbf{v}(x, y)$ with $y_0 < y < y_0'$.

Note that in order to determine the minimum $\mathbf{v}$-value in each class, we may restrict our attention to $\mathbf{v}$-values at local minima.

Lemma 4. *For each t, $0 \leq t \leq a - 1$, we have*

$$\begin{aligned} &\mathbf{v}\big(\{x + (a - \ell)(t + 1)\} \bmod a, t + 1\big) - \mathbf{v}\big(\{x + (a - \ell)t\} \bmod a, t\big) \\ &\quad = b(a - \ell) + c \text{ or } c - b\ell. \end{aligned}$$

Proof. This follows directly from the observation

$$\{x + (a - \ell)(t + 1)\} \bmod a - \{x + (a - \ell)t\} \bmod a = a - \ell \text{ or } -\ell. \quad \square$$

Remark 1. We note that Lemma 2 also follows directly from Lemma 4. We have

$$\begin{aligned}&\mathbf{v}\big(x+(a-\ell)(t+1) \bmod a, t+1\big)-\mathbf{v}\big(x+(a-\ell)t \bmod a, t\big)\\&\quad\geq c-b\ell\geq(k-\ell)b\geq 0.\end{aligned}$$

Hence $\mathbf{m}(bx)=bx$ for $1\leq x\leq a-1$, and so

$$\mathsf{g}(a,b,c)=\left(\max_{1\leq x\leq a-1} bx\right)-a=b(a-1)-a=\mathsf{g}(a,b).$$

In order to compare the $\mathbf{v}$-values at local minimum in the class (bx), we note that the list of integers in this class can be generated in one of two ways. Beginning with $\mathbf{v}(x,0)$, for each increment by 1 to the y-coordinate, we could *either* add $a-\ell$ to the x-coordinate *or* subtract ℓ from the x-coordinate; the result is the same since each coordinate may be assumed to be reduced modulo a. We call each such operation a *step*. Note that there is exactly one value of x mod a corresponding to a value of y. These sequences of steps give rise to two parallel methods of attack; we follow the first method of successively applying the transformation $(x,y)\to\big((x+a-\ell) \bmod a, y+1\big)$. Throughout the rest of this paper, we work with the first method but give the parallel result in the second case, typically without giving a proof. It is clear that if (x_0,y_0) is a local minimum, the next possible local minimum will occur precisely when its x-coordinate first reaches a or exceeds it. To make these comparisons possible, we employ the following parallel sets of notations.

Definition 2. We define nonnegative integers q, $\overline{q}$, r, $\overline{r}$ by

$$q:=\left\lfloor\tfrac{a}{a-\ell}\right\rfloor,\ r:=a-q(a-\ell);\quad \overline{q}:=\left\lfloor\tfrac{a}{\ell}\right\rfloor,\ \overline{r}:=a-\overline{q}\ell.$$

Thus $q(a-\ell)+r=a=\overline{q}\ell+\overline{r}$, with q, $\overline{q}\geq 1$, $0\leq r<a-\ell$ and $0\leq\overline{r}<\ell$.

Remark 2. Observe that $r=0$ implies $(a-\ell)\mid a$, and since $a\mid\big(c+b(a-\ell)\big)$, we also have $(a-\ell)\mid c$. Unless $\ell=a-1$, this contradicts our assumption that $\gcd(a,c)=1$. Therefore $r\neq 0$ unless $\ell=a-1$. In a similar manner, we note that $\overline{r}\neq 0$ unless $\ell=1$. The case $\ell=1$ is dealt with in Lemma 2 and the case $\ell=a-1$, due to Brauer & Shockley in [4], is dealt with in Corollary 1, as a special case of Theorem 3.

We next record the gap between successive local minima in terms of the notations just introduced. We give a proof for the first of these, but merely record the second since it only requires a parallel argument.

Lemma 5. *If* $\mathbf{v}(x_0,y_0)$, $\mathbf{v}(x_0',y_0')$ *are consecutive local minima, with* $0\leq x_0,x_0'<a-\ell$, *then*

$$(x_0', y_0') - (x_0, y_0) = \begin{cases} (a - \ell - r, q + 1) & \text{if } 0 \le x_0 < r; \\ (-r, q) & \text{if } r \le x_0 < a - \ell. \end{cases}$$

Proof. If $0 \le x_0 \le r - 1$, we need $q + 1$ steps to arrive at the next local minimum:

$$\begin{aligned} (x_0', y_0') &= \big(x_0 + (q + 1)(a - \ell) \bmod a, y_0 + (q + 1)\big) \\ &= \big(x_0 + (a - \ell - r), y_0 + (q + 1)\big). \end{aligned}$$

If $r \le x_0 \le a - \ell - 1$, we need q steps to arrive at the next local minimum:

$$(x_0', y_0') = \big(x_0 + q(a - \ell) \bmod a, y_0 + q\big) = (x_0 - r, y_0 + q). \quad \square$$

The steps that lead from one local minimum to the next are crucial to determining $\mathbf{m}(bx)$. Henceforth, we call the operation $(x, y) \to (x + a - \ell - r, y + q + 1)$ an $\uparrow$-step and the operation $(x, y) \to (x - r, y + q)$ a $\downarrow$-step. Note that an $\uparrow$-step applies when $0 \le x < r$ and results in an *increase* in the $\mathbf{v}$-value by $B := b(a - \ell - r) + c(q + 1)$ whereas a $\downarrow$-step applies when $r \le x < a - \ell$ and results in a *decrease* in the $\mathbf{v}$-value by $A := br - cq$.

Lemma 6. *If $\mathbf{v}(x_0, y_0)$, $\mathbf{v}(x_0', y_0')$ are consecutive local minima, with $0 \le x_0, x_0' < \ell$, then*

$$(x_0', y_0') - (x_0, y_0) = \begin{cases} (\overline{r}, \overline{q}) & \text{if } 0 \le x_0 < \ell - \overline{r}; \\ \big(-(\ell - \overline{r}), \overline{q} + 1\big) & \text{if } \ell - \overline{r} \le x_0 < \ell. \end{cases}$$

Analogous to the terminology following Lemma 5, we call the operation $(x, y) \to (x + \overline{r}, y + \overline{q})$ an $\uparrow$-step and the operation $(x, y) \to \big(x - (\ell - \overline{r}), y + \overline{q} + 1\big)$ a $\downarrow$-step. Note that an $\uparrow$-step applies when $0 \le x < \ell - \overline{r}$ and results in an *increase* in the $\mathbf{v}$-value by $\overline{B} := b\overline{r} + c\overline{q}$ whereas a $\downarrow$-step applies when $\ell - \overline{r} \le x < \ell$ and results in a *decrease* in the $\mathbf{v}$-value by $\overline{A} := b(\ell - \overline{r}) - c(\overline{q} + 1)$.

The computation of $\mathbf{g}(a, b, c)$ is greatly simplified by restricting the evaluation of the minimum integer in the class (bx) for all x to those x less than $a - \ell$, and by a parallel argument to those x less than ℓ.

Lemma 7. *Let $\ell > k$. For $1 \le x \le a - 1$, $\mathbf{m}(bx) = \min\big\{bx, \mathbf{m}(bx') + cy'\big\}$, where*

$$(x', y') = \begin{cases} \Big((x \bmod (a - \ell)) - r + a - \ell, q - \lfloor \frac{x}{a-\ell} \rfloor + 1\Big) \\ \quad \text{if } 0 \le x \bmod (a - \ell) \le r - 1; \\ \Big((x \bmod (a - \ell)) - r, q - \lfloor \frac{x}{a-\ell} \rfloor\Big) \\ \quad \text{if } r \le x \bmod (a - \ell) < a - \ell. \end{cases}$$

Proof. For $1 \le x \le a - 1$, write $x = \lfloor \frac{x}{a-\ell} \rfloor (a - \ell) + x \bmod (a - \ell)$. If $x \bmod (a - \ell) \le r - 1$, the first local minimum after $(x, 0)$ is achieved after $q - \lfloor \frac{x}{a-\ell} \rfloor + 1$ steps, and if

$x \bmod (a-\ell) > r$, that local minimum is achieved after $q - \lfloor \frac{x}{a-\ell} \rfloor$ steps. Thus the first local minimum after the initial $(x, 0)$ is at (x', y'), with x' and y' as given in the result. But now the remaining local minima are clearly those that can be achieved by starting at $(x', 0)$ but incrementing each y-coordinate by y'. □

Lemma 8. *Let $\ell > k$. For $1 \le x \le a-1$,*

$$\mathbf{m}(bx) = \mathbf{m}\big(b(x \bmod \ell)\big) + c \left\lfloor \tfrac{x}{\ell} \right\rfloor.$$

Proof. The proof is along similar lines to that in Lemma 7, but is easier and gives a neater result. For $1 \le x \le a-1$, write $x = \lfloor \frac{x}{\ell} \rfloor \ell + x \bmod \ell$. Starting with the initial $(x, 0)$, the $\mathbf{v}$-values decrease in each of the first $\lfloor \frac{x}{\ell} \rfloor$ steps, leading up to $\big(x \bmod \ell, \lfloor \frac{x}{\ell} \rfloor\big)$. Since the remaining local minima are again those that can be achieved by starting at $(x \bmod \ell, 0)$ but incrementing each y-coordinate by $\lfloor \frac{x}{\ell} \rfloor$, it follows that

$$\mathbf{m}(bx) = \min\Big\{bx, \mathbf{m}\big(b(x \bmod \ell)\big) + c \left\lfloor \tfrac{x}{\ell} \right\rfloor \Big\}.$$

Moreover,

$$\mathbf{m}\big(b(x \bmod \ell)\big) + c \left\lfloor \tfrac{x}{\ell} \right\rfloor \le b(x \bmod \ell) + c \left\lfloor \tfrac{x}{\ell} \right\rfloor \le bx$$

the second inequality since $bx - \big\{b(x \bmod \ell) + c \left\lfloor \frac{x}{\ell} \right\rfloor \big\} = (b\ell - c) \left\lfloor \frac{x}{\ell} \right\rfloor \ge 0$. This completes the proof. □

3.4. Formulae for the cases $\ell > k$, $br < cq$ and $\ell > k$, $b(\ell - \overline{r}) < c(\overline{q} + 1)$

Lemma 7 gives explicit formulae for $\mathsf{g}(a, b, c)$ in the case $br < cq$, and Lemma 8 for the parallel case $b(\ell - \overline{r}) < c(\overline{q} + 1)$. These are easily derived because it turns out that $\mathbf{m}(bx) = bx$ for all x precisely when the above stated inequalities hold. In particular, one can derive a simple symmetric formula for $\mathsf{g}(a, b, c)$ when $a \mid (b + c)$ from these results.

Theorem 3. *If $\ell > k$ and $br < cq$, then*

$$\mathsf{g}(a,b,c) + a = \begin{cases} b\big\{(\lambda+1)(a-\ell) + r - 1\big\} & \text{if } \lambda \ge \frac{c(q-1)-br}{b(a-\ell)+c}; \\ b(a-\ell-1) + c(q-\lambda-1) & \text{if } \lambda \le \frac{c(q-1)-br}{b(a-\ell)+c}, \end{cases}$$

where $\lambda := \left\lfloor \frac{cq-br}{b(a-\ell)+c} \right\rfloor$.

Proof. Let (x_0, y_0), (x_0', y_0') be consecutive local minima. From Lemma 5 we see that $\mathbf{v}(x_0', y_0') - \mathbf{v}(x_0, y_0) = b(a-\ell-r) + c(q+1)$ or $cq - br$, both of which are positive. Hence $\mathbf{m}(bx) = bx$ for $1 \le x < a-\ell$, and so $\mathbf{m}(bx) = \min\{bx, bx' + cy'\}$ for $1 \le x \le a-1$ and x', y' as given by Lemma 7.

Fix x, $1 \le x \le a-1$, and write $\lfloor \frac{x}{a-\ell} \rfloor = m$ and $x \bmod (a-\ell) = s$ in Lemma 7. Set $\epsilon = 0$ or 1 according as $s \ge r$ or $s < r$. From Lemma 7, $\mathbf{m}(bx) = bx$ if and only if

$$b\{(m-\epsilon)(a-\ell)+r\} \le c\{q-(m-\epsilon)\} \Leftrightarrow (m-\epsilon)\{b(a-\ell)+c\} \le cq - br \Leftrightarrow m \le \lambda + \epsilon.$$

Thus for fixed s, $0 \le s < a-\ell$,

$$\max_m \mathbf{m}(bx) = \max\left\{b\big((\lambda+\epsilon)(a-\ell)+s\big), b\big(\epsilon(a-\ell)+s-r\big)+c(q-\lambda-1)\right\}.$$

A little simplification shows that

$$\left\{b\big(\epsilon(a-\ell)+s-r\big)+c(q-\lambda-1)\right\} - \left\{b\big((\lambda+\epsilon)(a-\ell)+s\big)\right\} = c(q-1) - br - \lambda\big\{b(a-\ell)+c\big\},$$

and this is independent of s and ϵ. Since $b\big\{\lambda(a-\ell)+a-\ell-1\big\} < b\big\{(\lambda+1)(a-\ell)+r-1\big\}$, it follows that

$$\max_{0 \le x \le a-1} \mathbf{m}(bx) = \begin{cases} b\big\{(\lambda+1)(a-\ell)+r-1\big\} & \text{if } \lambda \ge \frac{c(q-1)-br}{b(a-\ell)+c}; \\ b(a-\ell-1)+c(q-\lambda-1) & \text{if } \lambda \le \frac{c(q-1)-br}{b(a-\ell)+c}. \end{cases}$$

To complete the proof, we note that $b\big\{(\lambda+1)(a-\ell)+r-1\big\} = b(a-\ell-1)+c(q-\lambda-1)$ if and only if $\lambda\big(b(a-\ell)+c\big) = c(q-1)-br$. □

Example 1. We compute $\mathsf{g}(113, 127, 157)$ by using Theorem 3. We have $k = 1$, $\ell = 100$, $q = 8$, $r = 9$, $\lambda = 0$, and $c(q-1) < br$. By the first case, $\mathsf{g}(113, 127, 157) = (127 \cdot 21) - 113 = 2554$.

Corollary 1. (Brauer & Shockley [4]). *If $a \mid (b+c)$, then*

$$\mathsf{g}(a,b,c) + a = \begin{cases} b\lfloor \frac{ac}{b+c} \rfloor & \text{if } \lfloor \frac{ac}{b+c} \rfloor \ge \frac{(a-1)c}{b+c}; \\ c\lfloor \frac{ab}{b+c} \rfloor & \text{if } \lfloor \frac{ac}{b+c} \rfloor \le \frac{(a-1)c}{b+c}. \end{cases}$$

Proof. Observe that $\gcd(b+c, b) = 1 = \gcd(b+c, c)$ since $\gcd(b, c) = 1$. Hence $(b+c) \nmid ab$, $(b+c) \nmid ac$, and so $\lfloor \frac{ab}{b+c} \rfloor + \lfloor \frac{ac}{b+c} \rfloor = a-1$. Now $a \mid (b+c)$ if and only $\ell = a-1$, and the result follows as a direct consequence of Theorem 3. □

Remark 3. Corollary 1 also admits a direct proof; see [30]. Since $a \mid (b+c)$ implies $a - \ell = 1$, the only two local minima in the class (bx) are at $(x, 0)$ and $(0, a-x)$. Thus, $\mathbf{m}(bx)$ equals bx if $x \le \frac{ac}{b+c}$ and $c(a-x)$ if $x \ge \frac{ac}{b+c}$. Hence $\mathsf{g}(a,b,c) = \max\big\{b\lfloor \frac{ac}{b+c} \rfloor, c(a - \lceil \frac{ac}{b+c} \rceil)\big\} - a$. The result now follows from the observation $\lfloor \frac{ab}{b+c} \rfloor + \lfloor \frac{ac}{b+c} \rfloor = a-1$ and $b+c$ divides neither ab nor ac.

Theorem 3 applies in the case $\ell > k$ and $br < cq$. Exact formula for $\mathsf{g}(a,b,c)$ in the parallel case $\ell > k$ and $b(\ell - \overline{r}) < c(\overline{q}+1)$ given in Theorem 4 requires a similar, and somewhat easier, argument. There are several instances where exactly one of $br < cq$, $b(\ell - \overline{r}) < c(\overline{q}+1)$ holds under $\ell > k$. The two theorems together therefore enlarge the scope of the results, and the simplicity of the formulae given by each of these two theorems makes this an even more attractive proposition.

Theorem 4. *Suppose $\ell > k$ and $b(\ell - \overline{r}) < c(\overline{q}+1)$. Then*

$$\mathsf{g}(a,b,c) + a = \begin{cases} b(\ell-1) + c(\overline{q}-1) & \text{if } 0 \le \overline{r} < \ell - k; \\ b(\overline{r}-1) + c\overline{q} & \text{if } \ell - k \le \overline{r} < \ell. \end{cases}$$

Proof. Let (x_0, y_0), (x_0', y_0') be consecutive local minima. From Lemma 6 we see that $\mathbf{v}(x_0', y_0') - \mathbf{v}(x_0, y_0) = b\overline{r} + c\overline{q}$ or $c(\overline{q}+1) - b(\ell - \overline{r})$, both of which are positive. Hence $\mathbf{m}(bx) = bx$ for $1 \le x < \ell$, so that by Lemma 8, for $1 \le x \le a-1$ we now have

$$\mathbf{m}(bx) = b(x \bmod \ell) + c\left\lfloor \tfrac{x}{\ell} \right\rfloor.$$

Therefore

$$\max_{1 \le x \le a-1} \mathbf{m}(bx) = \max\left\{b(\overline{r}-1) + c\overline{q}, b(\ell-1) + c(\overline{q}-1)\right\}.$$

To complete the proof, note that $b(\overline{r}-1)+c\overline{q} \ge b(\ell-1)+c(\overline{q}-1)$ if and only if $c \ge b(\ell-\overline{r})$ if and only if $k \ge \ell - \overline{r}$. $\square$

Example 2. We compute $\mathsf{g}(113, 127, 182)$ by using Theorem 4. We have $k = 1$, $\ell = 13$, $\overline{q} = 8$, and $\overline{r} = 9$. By the first case, $\mathsf{g}(113, 127, 182) = (127 \cdot 12) + (182 \cdot 7) - 113 = 2685$.

Corollary 2. (Selmer [24]). *If h is a positive integer and* $\gcd(a,d) = 1$, *then*

$$\mathsf{g}(a, ha+d, ha+2d) = ha\lfloor \tfrac{a-2}{2} \rfloor + (h-1)a + d(a-1).$$

Proof. If a is even, by Theorem 2,

$$\begin{aligned} \mathsf{g}(a, ha+d, ha+2d) &= 2 \cdot \mathsf{g}\big(ha+d, \tfrac{a}{2}, h\tfrac{a}{2}+d\big) + (ha+d) \\ &= 2 \cdot \mathsf{g}\big(\tfrac{a}{2}, h\tfrac{a}{2}+d\big) + (ha+d), \end{aligned}$$

since $ha + d \in \Gamma\big(\{\tfrac{a}{2}, h\tfrac{a}{2}+d\}\big)$. Now

$$\begin{aligned} 2 \cdot \mathsf{g}\big(\tfrac{a}{2}, h\tfrac{a}{2}+d\big) + (ha+d) &= a\big(h\tfrac{a}{2}+d\big) - a - (ha+2d) + (ha+d) \\ &= \tfrac{a}{2}(ha-2) + d(a-1), \end{aligned}$$

as desired.

If a is odd, the result follows directly from Theorem 4 with $k = 1$, $\ell = 2$, $\overline{q} = \frac{a-1}{2}$, $\overline{r} = 1$. $\square$

Remark 4. Corollary 2 is a special case of the following result of Selmer:

$$\mathsf{g}(a, ha + d, \ldots, ha + kd) = ha\lfloor \tfrac{a-2}{k} \rfloor + (h-1)a + d(a-1)$$

if h is any positive integer and $\gcd(a, d) = 1$. This result was also given by Tripathi [31], and generalizes a result of Roberts [20] about the Frobenius number for arithmetic progressions.

Corollary 3. (Einstein, Lichtblau, Strzebonski & Wagon [8]). *For any positive integer a,*

$$\mathsf{g}(a, a+1, a+4) = \begin{cases} \frac{1}{4}(a^2 + 8a - 4) & \text{if } a \equiv 0 \pmod 4; \\ \frac{1}{4}(a^2 + 7a - 8) & \text{if } a \equiv 1 \pmod 4; \\ \frac{1}{4}(a^2 + 6a - 12) & \text{if } a \equiv 2 \pmod 4; \\ \frac{1}{4}(a^2 + 5a - 4) & \text{if } a \equiv 3 \pmod 4. \end{cases}$$

Proof. If $a \equiv 0 \pmod 4$, by Theorem 2,

$$\begin{aligned} \mathsf{g}(a, a+1, a+4) &= 4 \cdot \mathsf{g}\left(\tfrac{a}{4}, \tfrac{a}{4} + 1, a+1\right) + 3(a+1) = 4 \cdot \mathsf{g}\left(\tfrac{a}{4}, \tfrac{a}{4} + 1\right) + 3(a+1) \\ &= \tfrac{1}{4}(a^2 + 8a - 4). \end{aligned}$$

Note that the second equality holds because $a+1 \in \Gamma\left(\frac{a}{4}, \frac{a}{4}+1\right)$ and recall that $\mathsf{g}(m, n) = mn - m - n$ when $\gcd(m, n) = 1$.

Suppose $a \equiv 1 \pmod 4$. Observe that the result holds for $a = 1$. If $a \equiv 2 \pmod 3$, by Theorem 2 and the first case of Corollary 1,

$$\begin{aligned} \mathsf{g}(a, a+1, a+4) &= 3 \cdot \mathsf{g}\left(\tfrac{a+1}{3}, \tfrac{a+1}{3} + 1, a\right) + 2a \\ &= \frac{1}{4}(a+4)(a-1) + a - 1 = \frac{1}{4}(a^2 + 7a - 8). \end{aligned}$$

If $a \not\equiv 2 \pmod 3$, the variables are pairwise coprime and $\ell = 4$, $k = 1$, $\overline{q} = \frac{a-1}{4}$ and $\overline{r} = 1$ if $a > 1$. Thus $b(\ell - \overline{r}) < c(\overline{q} + 1)$ holds, and from the first part of Theorem 4,

$$\mathsf{g}(a, a+1, a+4) = 3(a+1) + \tfrac{1}{4}(a+4)(a-5) - a = \tfrac{1}{4}(a^2 + 7a - 8).$$

Suppose $a \equiv 2 \pmod 4$. If $a \equiv 2 \pmod 3$, by Theorem 2 and the first case of Corollary 1,

$$\begin{aligned} \mathsf{g}(a, a+1, a+4) &= 3 \cdot \mathsf{g}\left(\tfrac{a+1}{3}, \tfrac{a+1}{3} + 1, a\right) + 2a \\ &= \frac{1}{4}(a+4)(a-2) + a - 1 = \frac{1}{4}(a^2 + 6a - 12). \end{aligned}$$

If $a \not\equiv 2 \pmod 3$, by Theorem 2,

$$\mathsf{g}(a, a+1, a+4) = 2 \cdot \mathsf{g}\left(\tfrac{a}{2}, \tfrac{a}{2}+2, a+1\right) + (a+1) \equiv 2 \cdot \mathsf{g}(b, b+2, 2b+1),$$

where $b = \frac{a}{2} \equiv 1 \pmod 2$ and $b \not\equiv 1 \pmod 3$. Then b, $b+2$, $2b+1$ are pairwise coprime and $\ell = \frac{b+1}{2}$, $k = 1$, $\overline{q} = 1$ and $\overline{r} = \frac{b-1}{2}$. Thus $b(\ell - \overline{r}) < c(\overline{q}+1)$ holds, and from the second part of Theorem 4,

$$\mathsf{g}(b, b+2, 2b+1) = \tfrac{1}{2}(b^2 + b - 4), \quad \mathsf{g}(a, a+1, a+4) = \tfrac{1}{4}(a^2 + 6a - 12).$$

Suppose $a \equiv 3 \pmod 4$. If $a \equiv 2 \pmod 3$, by Theorem 2 and the second case of Corollary 1,

$$\begin{aligned}\mathsf{g}(a, a+1, a+4) &= 3 \cdot \mathsf{g}\left(\tfrac{a+1}{3}, \tfrac{a+1}{3}+1, a\right) + 2a = \tfrac{1}{4}a(a+1) + a - 1 \\ &= \tfrac{1}{4}(a^2 + 5a - 4).\end{aligned}$$

If $a \not\equiv 2 \pmod 3$, the variables are pairwise coprime and $\ell = 4$, $k = 1$, $\overline{q} = \frac{a-3}{4}$ and $\overline{r} = 3$. Thus $b(\ell - \overline{r}) < c(\overline{q}+1)$ holds, and from the second part of Theorem 4,

$$\mathsf{g}(a, a+1, a+4) = 2(a+1) + \tfrac{1}{4}(a+4)(a-3) - a = \tfrac{1}{4}(a^2 + 5a - 4). \quad \square$$

Remark 5. Einstein et al. [8] remarked that the result in Corollary 3 followed from a more general result of Rødseth [22]. They also treated the general case $\mathsf{g}(a, a+1, a+4, \ldots, a+k^2)$ by using a geometric algorithm, and conjectured that this Frobenius number is of the form $\frac{1}{k^2}(a^2 + \alpha a) - \beta$ for some integers α, β which depend on k and the residue class of a modulo k^2. Kan et al. [14] gave an exact formula for $\mathsf{g}(a, a+1, a+d)$ when $2 \le d \le 5$ and $a > d(d-4)+1$, and also an upper bound for general d, although no proofs were given.

3.5. The cases $\ell > k$, $br > cq$ and $\ell > k$, $b(\ell - \overline{r}) > c(\overline{q}+1)$

Theorem 3 covers the case $br < cq$ and Theorem 4 the parallel case $b(\ell - \overline{r}) < c(\overline{q}+1)$, both under the assumption $\ell > k$. The assumption that $\gcd(b, c) = 1$ implies $br \neq cq$ and $b(\ell - \overline{r}) \neq c(\overline{q}+1)$ since $r < a - \ell < a < c$ and $\ell - \overline{r} < \ell < a < c$. So it remains to consider the remaining subcases for $\ell > k$, namely, the case $br > cq$ and the parallel case $b(\ell - \overline{r}) > c(\overline{q}+1)$. These turn out to be far more challenging since there exist $x < a - \ell$ for which $\mathbf{m}(bx) < bx$, and in the parallel case, $x < \ell$ for which $\mathbf{m}(bx) < bx$. We now further extend Lemma 7 and the parallel Lemma 8.

Lemma 9. *If $\ell > k$ and $br > cq$, then*

$$\max_{0 \le x \le a-1} \mathbf{m}(bx) = \max\left\{ \max_{0 \le x \le (a-\ell-1) \bmod r} \mathbf{m}(bx) + cq, \max_{(a-\ell) \bmod r \le x \le r-1} \mathbf{m}(bx) \right\} + cq \left\lfloor \frac{a-\ell-1}{r} \right\rfloor.$$

Proof. For $1 \le x \le a-1$, write $\lfloor \frac{x}{a-\ell} \rfloor = m$ and $x \bmod (a-\ell) = s$. From the proof of Theorem 3 we have $\mathbf{m}(bx) = \mathbf{m}(bx') + cy'$ if $m \ge 1$, since $br > cq$ implies $\lambda < 0$, where x', y' are as in Lemma 7. Hence by Lemma 7,

$$\begin{aligned}\max_{0\le x\le a-1} \mathbf{m}(bx) &= \max_{m\ge 1} \Big\{ \max_{0\le s\le a-\ell-r-1} \mathbf{m}(bs) + c(q-m), \\ &\qquad \max_{a-\ell-r\le s\le a-1} \mathbf{m}(bs) + c(q-m+1) \Big\} \\ &= \max \Big\{ \max_{0\le x\le a-\ell-r-1} \mathbf{m}(bx) + c(q-1), \\ &\qquad \max_{a-\ell-r\le x\le a-\ell-1} \mathbf{m}(bx) + cq \Big\}.\end{aligned}$$

Now suppose that $0 \le x \le a-\ell-1$. By Lemma 5, successive local minima starting at $(x, 0)$ are obtained by repeatedly applying the step $(x, y) \to (x-r, y+q)$ as long as the x-value remains greater than or equal to r. Each such step results in lowering the $\mathbf{v}$-value since $br > cq$. So, to each x, $0 \le x \le a-\ell-r-1$, there corresponds an x' with $a-\ell-r \le x' \le a-\ell-1$ and $\mathbf{m}(bx') > \mathbf{m}(bx)$. If mr is the unique multiple of r satisfying $a-\ell-r \le mr \le a-\ell-1$, then $m-1$ such steps can be applied for those $x < mr$ and m such steps for $x \ge mr$ in this interval. Hence

$$\begin{aligned}\max_{0\le x\le a-1} \mathbf{m}(bx) &= \max_{a-\ell-r\le x\le a-\ell-1} \mathbf{m}(bx) + cq \\ &= \max \Big\{ \max_{0\le x\le (a-\ell-1) \bmod r} \mathbf{m}(bx) + cq, \max_{(a-\ell) \bmod r\le x\le r-1} \mathbf{m}(bx) \Big\} \\ &\quad + cq \left\lfloor \frac{a-\ell-1}{r} \right\rfloor. \quad \square\end{aligned}$$

Lemma 10. *If $\ell > k$ and $b(\ell-\overline{r}) > c(\overline{q}+1)$, then*

$$\begin{aligned}&\max_{0\le x\le a-1} \mathbf{m}(bx) = \\ &\max \Big\{ \max_{0\le x\le (\overline{r}-1) \bmod (\ell-\overline{r})} \mathbf{m}(bx) + c(\overline{q}+1), \max_{\overline{r} \bmod (\ell-\overline{r})\le x\le \ell-\overline{r}-1} \mathbf{m}(bx) \Big\} \\ &\quad + c\left\{ (\overline{q}+1) \left\lfloor \frac{\ell-1}{\ell-\overline{r}} \right\rfloor - 2 \right\}.\end{aligned}$$

The proof of Lemma 10 follows along similar lines to that given for Lemma 9, using Lemmas 6 and 8, and is therefore omitted. Lemmas 9 and 10 reduce the problem of determining $\mathsf{g}(a, b, c)$ to comparing $\mathbf{m}(bx)$ only for $x < r$ (respectively, only for $x < \ell - \overline{r}$) when $br > cq$ (respectively, $b(\ell-\overline{r}) > c(\overline{q}+1)$). The assumptions imply that there exist $x < r$ (respectively, $x < \ell-\overline{r}$) such that $\mathbf{m}(bx) < bx$, and in fact, sometimes with $\mathbf{m}(bx) = cy$ for some $y \ge 1$. This in turn implies *either* $\mathbf{m}\big(bx\big) = \mathbf{m}\big(b(x-1)\big) + b$ *or* $\mathbf{m}\big(bx\big) = cy$ for some $y \ge 1$. Therefore the following definition is crucial to the remaining cases.

Definition 3. Let $\ell > k$ and $br > cq$. We define the set $\mathcal{X}$ by

$$\mathcal{X} := \{x : c \mid \mathbf{m}(bx), 0 < x \le r\}.$$

Remark 6. If $br > cq$, note that $\mathbf{m}(br) = cq$ since the only two local minima in the class (br) are at $(r, 0)$ and $(0, q)$. Hence $r \in \mathcal{X}$. Note also that $\hat{x} := \min \mathcal{X} = \min\{x : \mathbf{m}(bx) \ne bx\}$. For if $\overline{x} = \min\{x : \mathbf{m}(bx) \ne bx\}$, then $\mathbf{m}(b\overline{x}) = bx_0 + cy_0$ for some $y_0 \ge 1$. Hence $bx_0 < bx_0 + cy_0 < b\overline{x}$, and so that $x_0 < \overline{x}$. But then $cy_0 \in \big(b(\overline{x} - x_0)\big)$ and $cy_0 < b(\overline{x} - x_0)$, so that $\mathbf{m}\big(b(\overline{x} - x_0)\big) < b(\overline{x} - x_0)$. Thus $\overline{x} \in \mathcal{X}$, and hence $\hat{x} = \overline{x}$.

Definition 4. Set $A := br - cq$, $B := b(a - \ell - r) + c(q + 1)$, and

$$\Lambda := \left\lfloor \tfrac{r}{a-\ell-r} \right\rfloor, \quad \Delta := \left\lfloor \tfrac{A}{B} \right\rfloor, \quad \Lambda' := \left\lfloor \tfrac{a-\ell-r}{r} \right\rfloor, \quad \Delta' := \left\lfloor \tfrac{B}{A} \right\rfloor.$$

Lemma 11. *Let $\ell > k$ and $br > cq$. Then*

$$\begin{aligned}\mathcal{X} &= \left\{ r\left(\left\lfloor \tfrac{(a-\ell-r)t}{r} \right\rfloor + 1\right) - (a - \ell - r)t : 0 \le t \le \mu' \right\} \\ &= \left\{ r - (a - \ell - r)t \bmod r : 0 \le t \le \mu' \right\},\end{aligned}$$

where μ' is the largest nonnegative integer m such that $\lfloor \frac{mB}{A} \rfloor = \lfloor \frac{m(a-\ell-r)}{r} \rfloor$. Let $u \equiv a - \ell \pmod{r}$. If $\mu' < \lfloor \frac{r}{u} \rfloor$, then

$$\mathcal{X} = \{r - ut : 0 \le t \le \mu'\}.$$

In particular, if $\Lambda > \Delta$ or $\Delta' > \Lambda'$, then

$$\mathcal{X} = \{r - (a - \ell - r)t : 0 \le t \le \Delta\}.$$

Proof. Recall from Lemma 5 that to go from one local minimum to the next, we use $(x, y) \to (x + a - \ell - r, y + q + 1)$ when $0 \le x \le r - 1$ and $(x, y) \to (x - r, y + q)$ when $r \le x \le a - \ell - 1$. For convenience, we call the first an $\uparrow$-step and the second a $\downarrow$-step. Note that an $\uparrow$-step results in an *increase* in the $\mathbf{v}$-value by $b(a - \ell - r) + c(q + 1) = B$ and a $\downarrow$-step a *decrease* in the $\mathbf{v}$-value by $br - cq = A$.

Observe that $r \in \mathcal{X}$ by Remark 6. Now suppose $r \ne x \in \mathcal{X}$. Then $\mathbf{m}(bx) = cy$ for some $y \ge 1$. Since $(x, 0)$ and $(0, y)$ are both local minimum in the class (bx), we can reach $(0, y)$ from $(x, 0)$ by a sequence of t_1 $\downarrow$-steps and t_2 $\uparrow$-steps for some $t_1 \ge 1$ and $t_2 \ge 1$. Hence $x = rt_1 - (a - \ell - r)t_2$, $y = qt_1 + (q + 1)t_2$, and so the inequality $0 < x < r$ reduces to $\frac{r}{a-\ell-r}(t_1 - 1) < t_2 < \frac{r}{a-\ell-r} t_1$. Therefore every element $x \in \mathcal{X}$, $x < r$, is of the form $rt_1 - (a - \ell - r)t_2$ for some positive integers t_1, t_2 satisfying the inequality given above. We claim that in addition, $x \in \mathcal{X}$ if and only if $t_2 \le \mu'$, where μ' is the largest nonnegative integer m such that $\lfloor \frac{mB}{A} \rfloor = \lfloor \frac{m(a-\ell-r)}{r} \rfloor$.

Let $x \in \mathcal{X}$, so that $x = rt_1 - (a - \ell - r)t_2$, where t_1, t_2 are positive integers with $\frac{r}{a-\ell-r}(t_1 - 1) < t_2 < \frac{r}{a-\ell-r}t_1$. Then $\mathbf{v}(x, 0) = bx > cy = \mathbf{v}(0, y)$, with $y = qt_1 + (q + 1)t_2$, and this reduces to $At_1 > Bt_2$. Suppose first that $1 \le t_2 \le \mu'$, and consider any local minimum (x', y') in the class (bx). Suppose we reach $(0, y)$ from (x', y') in s_1 $\downarrow$-steps and s_2 $\uparrow$-steps. Then $x' = rs_1 - (a - \ell - r)s_2 > 0$, so that $s_1 \ge \lceil \frac{(a-\ell-r)s_2}{r} \rceil = \lceil \frac{Bs_2}{A} \rceil$ since $s_2 \le t_2 \le \mu'$. But then $\mathbf{v}(x', y') - \mathbf{v}(0, y) = As_1 - Bs_2 = A(s_1 - \frac{Bs_2}{A}) > 0$. Thus $\mathbf{m}(bx) = cy$ whenever $t_2 \le \mu'$ and $0 < x < r$. Now suppose that $t_2 > \mu'$. Then $t_1 \le \lfloor \frac{(a-\ell-r)t_2}{r} \rfloor + 1 \le \lfloor \frac{Bt_2}{A} \rfloor < \frac{Bt_2}{A}$, so that $\mathbf{v}(x, 0) < \mathbf{v}(0, y)$. Since $x = r - \big((a - \ell - r)t_2 - r(t_1 - 1)\big)$, the inequality can be replaced by the condition $x = r - (a - \ell - r)t_2 \bmod r$. This completes the result in the general case.

If $\mu' \le \lfloor \frac{r}{u} \rfloor$, then $(a - \ell - r)t \equiv ut \pmod r$ and $0 \le ut < r$ for $0 \le t \le \mu'$. Hence $\mathcal{X} = \{r - ut : 0 \le t \le \mu'\}$.

To complete the proof, we first show that $\mu' = \Delta$ if $\Lambda > \Delta$ or $\Delta' > \Lambda'$. Observe that $\lfloor \frac{\Delta B}{A} \rfloor = 0$. Hence $\lfloor \frac{mB}{A} \rfloor = 0 = \lfloor \frac{m(a-\ell-r)}{r} \rfloor$ for $0 \le m \le \Delta$, since $\frac{B}{A} > \frac{a-\ell-r}{r} > 0$. If $\Lambda > \Delta$, then $\lfloor \frac{(\Delta+1)B}{A} \rfloor = 1$ and $\lfloor \frac{(\Delta+1)(a-\ell-r)}{r} \rfloor \le \lfloor \frac{\Lambda(a-\ell-r)}{r} \rfloor = 0$. If $\Delta' > \Lambda'$, then $\mu' = 0$ by definition. In either case, $\mu' = \Delta$ since $\Delta' > \Lambda'$ implies $\Delta' > 0$ which in turn implies $\Delta = 0$.

Finally, we show that $\mu' = \Delta < \lfloor \frac{r}{u} \rfloor$ if $\Lambda > \Delta$ or $\Delta' > \Lambda'$. If $\Lambda > \Delta$, then $\Lambda' = 0$ (since $\Lambda > 0$), so that $u = (a - \ell - r) - \Lambda' r = a - \ell - r$ and $\mu' = \Delta < \Lambda = \lfloor \frac{r}{a-\ell-r} \rfloor = \lfloor \frac{r}{u} \rfloor$. If $\Delta' > \Lambda'$, then $u = (a - \ell - r) - \Lambda' r \le r$ since $\frac{a-\ell-r}{r} < \Lambda' + 1$. Hence $\mu' = \Delta = 0 < \lfloor \frac{r}{u} \rfloor$.

Hence, in either case $\mathcal{X} = \{r - (a - \ell - r)t : 0 \le t \le \Delta\}$. This completes the result in the special case. $\square$

Remark 7. If $u = 0$, then $r \mid a - \ell$. If $r = 1$, the condition $br > cq$ is not met. If $r > 1$, choose a prime divisor p of r. Then p divides $a - \ell$, hence a and c, so that $\gcd(a, c) > 1$, violating our assumption. Hence $u \ne 0$ under the given assumptions.

Remark 8. The equation $\mu' = \lfloor \frac{r}{u} \rfloor$ is never possible. From Lemma 11, the given condition implies both $\Lambda = \Delta$ and $\Lambda' = \Delta'$ must hold. Therefore neither $\frac{r}{a-\ell-r}$ nor $\frac{B}{A}$ is an integer, so that exactly one of the equal pairs must equal 0. Observe that $\lfloor \frac{m(a-\ell-r)}{r} \rfloor = 0$ if and only if $m \le \Lambda$, whereas $\lfloor \frac{mB}{A} \rfloor = 0$ if and only if $m \le \Delta$. Since $\Lambda = \Delta$, it follows that $\mu' \ge \Lambda$.

If $\Lambda' = \Delta' = 0$, then $u = a - \ell - r$, so that $\lfloor \frac{r}{u} \rfloor = \Lambda$. But $\mu' \ge \Lambda + 1$, since $\lfloor \frac{m(a-\ell-r)}{r} \rfloor = \lfloor \frac{mB}{A} \rfloor = 1$ when $m = \Lambda + 1 = \Delta + 1$.

If $\Lambda = \Delta = 0$, write $\frac{B}{A} = \Delta' + \{\frac{B}{A}\}$ and $\frac{a-\ell-r}{r} = \Lambda' + \{\frac{a-\ell-r}{r}\} = \Delta' + \frac{u}{r}$, where $\{x\}$ denotes the fractional part of x. Since $\mu' + 1$ is the smallest positive integer m for which $\lfloor \frac{m(a-\ell-r)}{r} \rfloor < \lfloor \frac{mB}{A} \rfloor$ holds, we must have $(\mu' + 1)\frac{u}{r} < 1 \le (\mu' + 1)\{\frac{B}{A}\}$. Hence $\lfloor \frac{r}{u} \rfloor \ge \mu' + 1$.

3.6. Formulae for the cases $\ell > k$, $br > cq$ and $\ell > k$, $b(\ell - \overline{r}) > c(\overline{q} + 1)$

Theorem 5. *Let $\ell > k$ and $br > cq$. Let $u \equiv a - \ell \pmod r$, and let μ' be the largest nonnegative integer m such that $\lfloor \frac{mB}{A} \rfloor = \lfloor \frac{m(a-\ell-r)}{r} \rfloor$.*

(a) *If $\mu' < \lfloor \frac{r}{u} \rfloor$,*

$$\begin{aligned} &\mathrm{g}(a,b,c) + a \\ &\quad = \max\Big\{b\big(r - \mu' u - 1\big), b(u-1) + c\big(\mu'(q+1) + \big(\lfloor \tfrac{(a-\ell-r)\mu'}{r} \rfloor + 1\big)q\big)\Big\} \\ &\qquad + cq\lfloor \tfrac{a-\ell-1}{r} \rfloor. \end{aligned}$$

In particular, if $\Lambda > \Delta$

$$\begin{aligned} &\mathrm{g}(a,b,c) + a \\ &\quad = \max\Big\{b\big(r - \Delta(a-\ell-r) - 1, b(a-\ell-r-1) + c\big(\Delta(q+1) + q\big)\big)\Big\} + cq, \end{aligned}$$

and if $\Delta' > \Lambda'$

$$\mathrm{g}(a,b,c) + a = \max\Big\{b(r-1), b\big((a-\ell-1) \bmod r\big) + cq\Big\} + cq\lfloor \tfrac{a-\ell-1}{r} \rfloor.$$

(b) *Let $\mu' > \lfloor \frac{r}{u} \rfloor$. Let $\mathcal{X} = \{x_i : 0 \le i \le \mu'\}$, where*

$$x_i = r\big(\lfloor \tfrac{(a-\ell-r)i}{r} \rfloor + 1\big) - (a-\ell-r)i.$$

Set $y_i = q\big(\lfloor \frac{(a-\ell-r)i}{r} \rfloor + 1\big) + (q+1)i$ for $0 \le i \le \mu'$. Let $d_1 = \lceil \frac{r}{u} \rceil u - r$, and $d_2 = \hat{x} = \min \mathcal{X}$. Let p_i be the largest positive integer such that $x_{p_i} + d_i \in \mathcal{X}$ for $i = 1, 2$. Then

$$\mathrm{g}(a,b,c) + a = \max\big\{b\big(d_1 - 1\big) + cy_{p_1}, b\big(d_2 - 1\big) + cy_{p_2}\big\} + cq\left\lfloor \tfrac{a-\ell-1}{r} \right\rfloor.$$

Proof. (a) If $\mu' < \lfloor \frac{r}{u} \rfloor$, then $\mathcal{X} = \{r - ui : 0 \le i \le \mu'\}$ by Lemma 11. For $0 \le i \le \mu'$, let $x_i = r - ui = r\big(\lfloor \frac{(a-\ell-r)i}{r} \rfloor + 1\big) - (a-\ell-r)i$ and $\mathbf{m}(bx_i) = cy_i$. Since we require i $\uparrow$-steps and $\lfloor \frac{(a-\ell-r)i}{r} \rfloor + 1$ $\downarrow$-steps to arrive at $(0, y_i)$ from $(x_i, 0)$, we have $y_i = (q+1)i + q\big(\lfloor \frac{(a-\ell-r)i}{r} \rfloor + 1\big)$.

If $x < \hat{x} = \min \mathcal{X}$, then $\mathbf{m}(bx) = bx$ by Definition 3 and Remark 6. Any $x \notin X$, $\hat{x} < x < r$ is of the form $x_i + x'$ with $1 \le i \le \mu'$ and $0 < x' < u$. The sequence of local minimum starting with $(x, 0)$ consisting of i $\uparrow$-steps and y_i $\downarrow$-steps leads to the local minimum (x', y_i). Now $x' \le u - 1 \le \hat{x} - 1 = r - \mu' u - 1$, since $\mu' + 1 \le \frac{r}{u}$. So the sequence of local minimum after (x', y_i) results in numbers larger than $\mathbf{v}(x', y_i)$, which therefore equals $\mathbf{m}(bx)$. By Lemma 9,

$$\begin{aligned}\mathsf{g}(a,b,c)+a &= \max\Big\{\max_{0\le x\le u-1}\mathbf{m}(bx)+cq,\ \max_{u\le x\le r-1}\mathbf{m}(bx)\Big\}+cq\lfloor\tfrac{a-\ell-1}{r}\rfloor\\ &= \max\{\mathbf{v}(\hat{x}-1,0),\mathbf{v}(u-1,y_{\mu'})\}+cq\big\lfloor\tfrac{a-\ell-1}{r}\big\rfloor\\ &= \max\Big\{b\big(r-\mu'u-1\big),b(u-1)+c\big(\mu'(q+1)\\ &\quad+\big(\lfloor\tfrac{(a-\ell-r)\mu'}{r}\rfloor+1\big)q\big)\Big\}\\ &\quad+cq\lfloor\tfrac{a-\ell-1}{r}\rfloor.\end{aligned}$$

From Lemma 11, we know that $\Delta=\mu'<\lfloor\frac{r}{u}\rfloor$ when $\Lambda>\Delta$ or when $\Delta'>\Lambda'$. If $\Lambda>\Delta$, then $u=a-\ell-r$ and $\lfloor\frac{a-\ell-1}{r}\rfloor=1$. If $\Delta'>\Lambda'$, then $\Delta=0$ (since $\Delta'>0$). The result now follows from the general case.

(b) Suppose $\mu'>\lfloor\frac{r}{u}\rfloor$. Note that this implies that the elements in $\mathcal{X}$ do not have a fixed common difference. In fact, the difference between consecutive integers in $\mathcal{X}$ is either $d_1=\lceil\frac{r}{u}\rceil u-r$ or $d_2=\hat{x}$, with $d_1<d_2$ as we indirectly show in the following argument. Note also that the assumption also implies both $\Lambda=\Delta$ and $\Lambda'=\Delta'$; the converse is not true, as Examples 4 and 5 demonstrate.

Recall that $\mathbf{m}(bx)=bx$ for all $x<\hat{x}=\min\mathcal{X}$ by Remark 6, and $\mathbf{m}(bx)=cy$ for $x\in\mathcal{X}$ by Definition 3. For $x\notin\mathcal{X}$, $x>\hat{x}$, choose the largest element $x_j\in\mathcal{X}$ such that $x_j<x$ and write $x=x_j+x'$; this is possible because $\mu'\neq 0$. Let $\mathbf{m}(bx_j)=cy_j$. Applying the sequence of $\uparrow$-steps and $\downarrow$-steps that lead $(x_j,0)$ to $(0,y_j)$ must then lead $(x,0)$ to (x',y_j) through local minima, since otherwise the same sequence would lead some $(\tilde{x},0)$ to $(0,\tilde{y})$ with $x_j<\tilde{x}<x$, with the conclusion that $\mathbf{m}(b\tilde{x})=c\tilde{y}$ contradicting the definition of $x_j\in\mathcal{X}$. Therefore $\mathbf{m}(bx)=\mathbf{m}(bx_j)+b(x-x_j)=b(x-x_j)+cy_j$. By Lemma 9

$$\begin{aligned}\max_{0\le x\le a-1}\mathbf{m}(bx)=\max\Bigg\{&\max_{\substack{0\le x\le(a-\ell-1)\bmod r\\ x\in\mathcal{X}}}\mathbf{m}\big(b(x-1)\big)+cq,\\ &\max_{\substack{(a-\ell)\bmod r\le x\le r-1\\ x\in\mathcal{X}}}\mathbf{m}\big(b(x-1)\big)\Bigg\}\\ &+cq\left\lfloor\tfrac{a-\ell-1}{r}\right\rfloor.\end{aligned}$$

Write $\mathcal{X}=\{x_0,\dots,x_{\mu'}\}$, where $x_i=r\big(\lfloor\frac{(a-\ell-r)i}{r}\rfloor+1\big)-(a-\ell-r)i$. Note that $\mathbf{m}(bx_i)=cy_i$ where $y_i=q\big(\lfloor\frac{(a-\ell-r)i}{r}\rfloor+1\big)+(q+1)i$. Observe that if we arrange the elements in $\mathcal{X}$ in increasing order, starting with $\hat{x}=x_j$ and ending with $r=x_0$, the difference between consecutive elements is always one of two integers, $d_1=\lceil\frac{r}{u}\rceil u-r$ and $d_2=\hat{x}$. Now choose the largest positive integers p_1, p_2 such that $x_{p_i}+d_i\in\mathcal{X}$ for $i=1,2$; clearly one of p_1, p_2 is μ'. Hence the maximum above is reduced to choosing the larger of the values $\mathbf{m}(bx)$ corresponding to $x_1=x_{p_1}+d_1-1$ and $x_2=x_{p_2}+d_2-1$. Therefore

$$\mathsf{g}(a,b,c)+a=\max\left\{\mathbf{m}\big(bx_1\big),\mathbf{m}\big(bx_2\big)\right\}+cq\left\lfloor\tfrac{a-\ell-1}{r}\right\rfloor.$$

This completes the proof. □

Example 3. We compute $\mathsf{g}(100,101,159)$ by using Theorem 5 (a). We have $k=1$, $\ell=59$, $q=2$, $r=18$, $a-\ell-r=23$, $\Lambda=\Delta=0$ and $\Lambda'=\Delta'=1$. We also have $A=1500$, $B=2800$, $\mu'=1$, $u=5$, and $\lfloor\frac{r}{u}\rfloor=3$. By the general case, $\mathsf{g}(100,101,159)=\max\{\mathbf{v}(12,0),\mathbf{v}(4,7)\}+\mathbf{v}(0,4)-100=\mathbf{v}(4,11)-100=(101\cdot4)+(159\cdot11)-100=2053$.

Example 4. We compute $\mathsf{g}(133,172,199)$ by using Theorem 5 (a). We have $k=1$, $\ell=104$, $q=4$, $r=17$, $a-\ell-r=12$, $\Lambda=1$, $\Delta=0$ and $\Lambda'=0$, $\Delta'=1$. So both special cases apply, and each gives $\mathsf{g}(133,172,199)=\max\{\mathbf{v}(16,0),\mathbf{v}(11,4)\}+\mathbf{v}(0,4)-133=\mathbf{v}(16,4)-133=(172\cdot16)+(199\cdot4)-133=3415$.

Example 5. We compute $\mathsf{g}(100,101,139)$ by using Theorem 5 (b). We have $k=1$, $\ell=39$, $q=1$, $r=39$, $a-\ell-r=22$, $\Lambda=\Delta=1$ and $\Lambda'=\Delta'=0$. We also have $A=3800$, $B=2500$, $\mu'=4$, $u=22$, and $\lfloor\frac{r}{u}\rfloor=1$. Again $\mathcal{X}=\{12,17,29,34,39\}$, $d_1=5$, $d_2=12$, $p_1=4$, $p_2=1$, $x_{p_1}=29$, $y_{p_1}=11$, $x_{p_2}=17$, $y_{p_2}=3$. Hence $\mathsf{g}(100,101,137)=\max\{\mathbf{v}(4,11),\mathbf{v}(11,3)\}+\mathbf{v}(0,1)-100=\mathbf{v}(4,12)-100=(101\cdot4)+(139\cdot12)-100=1972$.

To describe the parallel case, when $\ell>k$ and $b(\ell-\overline{r})>c(\overline{q}+1)$, it is natural to define the set $\mathcal{X}$ somewhat differently:

$$\overline{\mathcal{X}}:=\big\{x:c\mid\mathbf{m}(bx),0<x\le\ell-\overline{r}\big\}.$$

These naturally give rise to remarks analogous to those in Remark 6 and the following definition.

Definition 5. Set $\overline{A}:=b(\ell-\overline{r})-c(\overline{q}+1)$, $\overline{B}:=b\overline{r}+c\overline{q}$, and

$$\overline{\Lambda}:=\left\lfloor\tfrac{\ell-\overline{r}}{\overline{r}}\right\rfloor,\quad\overline{\Delta}:=\left\lfloor\tfrac{\overline{A}}{\overline{B}}\right\rfloor,\quad\overline{\Lambda}':=\left\lfloor\tfrac{\overline{r}}{\ell-\overline{r}}\right\rfloor,\quad\overline{\Delta}':=\left\lfloor\tfrac{\overline{B}}{\overline{A}}\right\rfloor.$$

Lemma 12. *Let $\ell>k$ and $b(\ell-\overline{r})>c(\overline{q}+1)$. Then*

$$\begin{aligned}\overline{\mathcal{X}}&=\left\{(\ell-\overline{r})\left(\left\lfloor\tfrac{\overline{r}t}{\ell-\overline{r}}\right\rfloor+1\right)-\overline{r}t:0\le t\le\overline{\mu}'\right\}\\&=\left\{(\ell-\overline{r})-\big(\overline{r}t\bmod(\ell-\overline{r})\big):0\le t\le\overline{\mu}'\right\},\end{aligned}$$

where $\overline{\mu}'$ is the largest nonnegative integer m such that $\lfloor\frac{m\overline{B}}{\overline{A}}\rfloor=\lfloor\frac{m\overline{r}}{\ell-\overline{r}}\rfloor$.
Let $\overline{u}\equiv\overline{r}\,(\mathrm{mod}\ \ell-\overline{r})$. If $\overline{\mu}'\le\lfloor\frac{\ell-\overline{r}}{\overline{u}}\rfloor$, then

$$\overline{\mathcal{X}}=\big\{(\ell-\overline{r})-\overline{u}t:0\le t\le\overline{\mu}'\big\}.$$

In particular, if $\overline{\Lambda} > \overline{\Delta}$ *or* $\overline{\Delta}' > \overline{\Lambda}'$*, then*

$$\overline{\mathcal{X}} = \left\{\ell - \overline{r}(t+1) : 0 \le t \le \overline{\Delta}\right\}.$$

The proof of Lemma 12 follows on lines similar to the one for Lemma 11. We use Lemma 6 to go from one local minimum to the next, and call $(x, y) \to (x + \overline{r}, y + \overline{q})$ (when $0 \le x < \ell - \overline{r}$) an $\uparrow$-step and $(x, y) \to \big(x - (\ell - \overline{r}), y + \overline{q} + 1\big)$ (when $\ell - \overline{r} \le x < \ell$) a $\downarrow$-step. Note that an $\uparrow$-step results in an *increase* in the **v**-value by $b\overline{r} + c\overline{q} = \overline{B}$ and a $\downarrow$-step in a *decrease* in the **v**-value by $b(\ell - \overline{r}) - c(\overline{q} + 1) = \overline{A}$. We omit the details of the proof.

Remark 9. If $\overline{u} = 0$, then $(\ell - \overline{r}) \mid \overline{r}$. If $\ell - \overline{r} = 1$, the condition $b(\ell - \overline{r}) > c(\overline{q} + 1)$ is not met. If $\ell - \overline{r} > 1$, choose a prime divisor p of $\ell - \overline{r}$. Then p divides $\overline{r}$, hence ℓ, a and c, so that $\gcd(a, c) > 1$, violating our assumption. Hence $\overline{u} \ne 0$ under the given assumptions.

Remark 10. The equation $\overline{\mu}' = \lfloor \frac{\ell - \overline{r}}{\overline{u}} \rfloor$ is possible. For example, $a = 137$, $b = 251$, $c = 256$ give $\ell = 108$, $\overline{q} = 1$, $\overline{r} = 29$, $\ell - \overline{r} = 79$, $\overline{u} = 29$, $\overline{A} = 19317$, $\overline{B} = 7535$, so that $\overline{\mu}' = 2 = \lfloor \frac{\ell - \overline{r}}{\overline{u}} \rfloor$.

Theorem 6. *Let* $\ell > k$ *and* $b(\ell - \overline{r}) > c(\overline{q} + 1)$*. Let* $\overline{u} \equiv \overline{r} \pmod{\ell - \overline{r}}$*, and let* $\overline{\mu}'$ *be the largest nonnegative integer* m *such that* $\lfloor \frac{m\overline{B}}{\overline{A}} \rfloor = \lfloor \frac{m\overline{r}}{\ell - \overline{r}} \rfloor$*.*

(a) *If* $\overline{\mu}' \le \lfloor \frac{\ell - \overline{r}}{\overline{u}} \rfloor$*,*

$$\begin{aligned} &\mathsf{g}(a, b, c) + a \\ &\quad = \max\left\{b\big(\ell - \overline{r} - \overline{\mu}'\overline{u} - 1\big), b(\overline{u} - 1) + c\big(\overline{\mu}'\overline{q} + \big(\lfloor \tfrac{\overline{r}\overline{\mu}'}{\ell - \overline{r}} \rfloor + 1\big)(\overline{q} + 1)\big)\right\} \\ &\qquad + c\Big((\overline{q} + 1)\lfloor \tfrac{\ell - 1}{\ell - \overline{r}} \rfloor - 2\Big). \end{aligned}$$

In particular, if $\overline{\Lambda} > \overline{\Delta}$

$$\mathsf{g}(a, b, c) + a = \max\left\{b(\overline{r} - 1) + c\overline{q}(\overline{\Delta} + 2), b\big(\ell - (\overline{\Delta} + 1)\overline{r} - 1\big) + c(\overline{q} - 1)\right\},$$

and if $\overline{\Delta}' > \overline{\Lambda}'$

$$\begin{aligned} \mathsf{g}(a, b, c) + a = \max&\left\{b\big(\overline{r} - 1 \bmod (\ell - \overline{r})\big) + c(\overline{q} + 1), b(\ell - \overline{r} - 1)\right\} \\ &+ c\Big((\overline{q} + 1)\lfloor \tfrac{\ell - 1}{\ell - \overline{r}} \rfloor - 2\Big). \end{aligned}$$

(b) *Let* $\overline{\mu}' > \lfloor \frac{\ell - \overline{r}}{\overline{u}} \rfloor$*. Let* $\overline{\mathcal{X}} = \{x_i : 0 \le i \le \overline{\mu}'\}$*, where*

$$x_i = (\ell - \overline{r})\big(\lfloor \tfrac{\overline{r}i}{\ell - \overline{r}} \rfloor + 1\big) - \overline{r}i.$$

Set $y_i = (\overline{q}+1)\big(\lfloor \frac{\overline{r}i}{\ell-\overline{r}} \rfloor + 1\big) + \overline{q}i$ *for* $0 \le i \le \overline{\mu}'$. *Let* $\overline{u} = \overline{r} \pmod{\ell - \overline{r}}$, $d_1 = \lceil \frac{\ell-\overline{r}}{\overline{u}} \rceil \overline{u} - (\ell - \overline{r})$, *and* $d_2 = \hat{x} = \min \overline{\mathcal{X}}$. *Let* p_i *be the largest positive integer such that* $x_{p_i} + d_i \in \overline{\mathcal{X}}$ *for* $i = 1, 2$. *Then*

$$\mathsf{g}(a,b,c) + a = \max\left\{b\big(d_1 - 1\big) + cy_{p_1}, b\big(d_2 - 1\big) + cy_{p_2}\right\} + c\left\{(\overline{q}+1)\left\lfloor \tfrac{\ell-1}{\ell-\overline{r}} \right\rfloor - 2\right\}.$$

The proof of Theorem 6 follows on lines analogous to those for Theorem 5. The ↑-step $(x, y) \to (x+\overline{r}, y+\overline{q})$ replaces the ↑-step $(x, y) \to (x+a-\ell-r, y+q+1)$, and results in an increase in the **v**-value by $\overline{B}$ instead of B. Similarly the ↓-step $(x, y) \to (x-\ell+\overline{r}, y+\overline{q}+1)$ replaces the ↓-step $(x, y) \to (x - r, y + q)$, and results in a decrease in the **v**-value by $\overline{A}$ instead of A. The definitions of Definition 4 as well as the result of Lemma 11 carry over to analogous one given by Definition 3 and Lemma 12. One of the sets $\mathcal{X}$, $\overline{\mathcal{X}}$ is contained in the other; in particular, they have the same smallest element. We omit the details of the proof.

Example 6. We compute $\mathsf{g}(100, 101, 139)$ by using Theorem 6 (a). We have $k = 1$, $\ell = 39$, $\overline{q} = 2$, $\overline{r} = 22$, $\ell - \overline{r} = 17$, $\overline{\Lambda} = \overline{\Delta} = 0$ and $\overline{\Lambda}' = \overline{\Delta}' = 1$. We also have $\overline{A} = 1300$, $\overline{B} = 2500$, $\overline{\mu}' = 1$, $\overline{u} = 5$, and $\lfloor \frac{\ell-\overline{r}}{\overline{u}} \rfloor = 3$. By the general case, $\mathsf{g}(100, 101, 139) = \max\{\mathbf{v}(11, 0), \mathbf{v}(4, 8)\} + \mathbf{v}(0, 4) - 100 = \mathbf{v}(4, 12) - 100 = (101 \cdot 4) + (139 \cdot 12) - 100 = 1972$.

Example 7. We compute $\mathsf{g}(110, 151, 201)$ by using Theorem 6 (a). We have $k = 1$, $\ell = 21$, $\overline{q} = 5$, $\overline{r} = 5$, $\ell - \overline{r} = 16$, $\overline{\Lambda} = 3$, $\overline{\Delta} = 0$, and $\overline{\Lambda}' = 0$, $\overline{\Delta}' = 1$. So both special cases apply, and each gives $\mathsf{g}(110, 151, 201) = \max\{\mathbf{v}(4, 10), \mathbf{v}(15, 4)\} - 110 = \mathbf{v}(15, 4) - 110 = (151 \cdot 15) + (201 \cdot 4) - 110 = 2959$.

Example 8. We compute $\mathsf{g}(110, 151, 211)$ by using Theorem 6 (b). We have $k = 1$, $\ell = 91$, $\overline{q} = 1$, $\overline{r} = 19$, $\ell - \overline{r} = 72$, $\overline{\Lambda} = \overline{\Delta} = 3$ and $\overline{\Lambda}' = \overline{\Delta}' = 0$. We also have $\overline{A} = 10450$, $\overline{B} = 3080$, $\overline{\mu}' = 6$, $\overline{u} = 19$, and $\lfloor \frac{\ell-\overline{r}}{\overline{u}} \rfloor = 3$. Again $\overline{\mathcal{X}} = \{15, 30, 34, 49, 53, 68, 72\}$, $d_1 = 4$, $d_2 = 15$, $p_1 = 6$, $p_2 = 3$, $x_{p_1} = 30$, $y_{p_1} = 10$, $x_{p_2} = 15$, $y_{p_2} = 5$. Hence $\mathsf{g}(110, 151, 211) = \max\{\mathbf{v}(3, 10), \mathbf{v}(14, 5)\} + \mathbf{v}(0, 0) - 110 = \mathbf{v}(14, 5) - 110 = (151 \cdot 14) + (211 \cdot 5) - 110 = 3059$.

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