NOTES

Edited by: **John Duncan**

The Coin Exchange Problem for Arithmetic Progressions

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We consider a problem which, in the general case, dates back to Frobenius, and can be thought of as exchanging coins of arbitrary denomination with an infinite supply of coins of certain fixed denominations.

Hence, we are given integers $a_0, \dots, a_{k-1}$, and N, and we seek nonnegative integers $x_0, \dots, x_{k-1}$ such that

$$a_0 x_0 + \cdots + a_{k-1} x_{k-1} = N. \tag{1}$$

In this note, we will consider the case where the given coin denominations a_j are in arithmetic progression. We write $a_j = a + (j-1)d$, with $(a, d) = 1$, for $1 \leq j \leq k$. For a fixed value of $k \geq 2$, we denote by $g(a, d; k)$ (respectively, $n(a, d; k)$) the largest (respectively, number of) N such that

$$a\left(\sum_{j=0}^{k-1} x_j\right) + d\left(\sum_{j=0}^{k-1} j x_j\right) = N \tag{2}$$

has no solution in nonnegative integers.

We present a simple argument that results in a formula for $g(a, d; k)$ and $n(a, d; k)$. A variation of the formula for $g(a, 1; k)$ is due to [**Bra42**] while that for $n(a, 1; k)$ is due to [**NW72**]. [**Rob56**] generalized this to obtain $g(a, d; k)$, later simplified by [**Bat58**], and [**Gra73**] obtained the result for $n(a, d; k)$.

Let $\mathscr{C}$ denote a non zero residue class modulo a, and let $m_{\mathscr{C}}$ denote the least positive integer of the form $a(\sum_{j=0}^{k-1} x_j) + d(\sum_{j=0}^{k-1} j x_j)$ in $\mathscr{C}$. It is well known that both $g(a, d; k)$ and $n(a, d; k)$ can be readily derived from these minima. A derivation of these may be found in [**Tri89**], but we present it here for the sake of completeness.

Lemma 1.

(i) $g(a, d; k) = \max_{\mathscr{C}} m_{\mathscr{C}} - a$, *the maximum taken over all nonzero classes* $\mathscr{C}$.
(ii) $n(a, d\text{:} k) = 1/a \sum_{\mathscr{C}} m_{\mathscr{C}} - (a-1)/2$, *the sum taken over all nonzero classes* $\mathscr{C}$.

Proof: (i) Since $\max_{\mathscr{C}} m_{\mathscr{C}}$ and $\max_{\mathscr{C}} m_{\mathscr{C}} - a$ are in the same residue class modulo a, and $\max_{\mathscr{C}} m_{\mathscr{C}}$ is the least positive integer of the form (2) in its class, $g(a, d; k) \geq \max_{\mathscr{C}} m_{\mathscr{C}} - a$. On the other hand, if $N > \max_{\mathscr{C}} m_{\mathscr{C}} - a$, then $N \geq \max_{\mathscr{C}} m_{\mathscr{C}}$ for each class $\mathscr{C}$, so that N is of the given form.

(ii) The number of positive integers in class $\mathscr{C}$ which cannot be represented in the form given by (2) is $[m_{\mathscr{C}}/a]$, since $m_{\mathscr{C}}$ is the least representable integer in class $\mathscr{C}$. Hence, the total number of nonrepresentable integers is

$$\sum_{\mathscr{C}}\left[\frac{m_{\mathscr{C}}}{a}\right]=\sum_{\mathscr{C}}\frac{m_{\mathscr{C}}}{a}-\sum_{i=0}^{a-1}\frac{i}{a}=\frac{1}{a}\sum_{\mathscr{C}}m_{\mathscr{C}}-\frac{a-1}{2}. \quad \square$$

We note that the lemma together with its proof carries over to the general case given by (1), with g and n being defined analogously.

Lemma 2. *For each y, $1 \le y \le a-1$, the least positive integer of the form given by (2) in the class $dy \bmod a$ is given by*

$$a\left(1+\left[\frac{y-1}{k-1}\right]\right)+dy.$$

Proof: Each nonzero class $\mathscr{C}$ modulo a determines a unique y, with $1 \le y \le a-1$, such that $dy \in \mathscr{C}$. This clearly is the smallest multiple of d in the class $\mathscr{C}$. For this choice of y, we wish to minimize $\sum_{j=0}^{k-1}x_j$ in order to determine the smallest member of $\mathscr{C}$. With $y=(k-1)q+r$, $0 \le r \le k-2$, we may choose $x_{k-1}=q$, with $x_r=1$, other $x_j=0$ provided $r \ne 0$ but with all other $x_j=0$ if $r=0$. Thus, the minimum value for $\sum_{j=0}^{k-1}x_j$ is $q+1$ if $r \ne 0$ and q if $r=0$. This may be written more briefly as

$$\left(1+\left[\frac{y-1}{k-1}\right]\right),$$

so that

$$m_{\mathscr{C}}=a\left(1+\left[\frac{y-1}{k-1}\right]\right)+dy, \quad \text{where } dy \in \mathscr{C}. \quad \square$$

Theorem 1.

(i) $g(a,d;k)=a[(a-2)/(k-1)]+d(a-1)$;
(ii) $n(a,d;k)=\frac{1}{2}(a+t)(1+[(a-2)/(k-1)])+\frac{1}{2}(a-1)(d-1)$, *where t is the smallest nonnegative integer such that $a-2 \equiv t \pmod{k-1}$.*

Proof: The theorem is an easy consequence of the two lemmas.

(i)
$$\begin{aligned} g(a,d;k) &= max_{\mathscr{C}}m_{\mathscr{C}}-a \\ &= max_{1\le y\le a-1}a\left[\frac{y-1}{k-1}\right]+dy \\ &= a\left[\frac{a-2}{k-1}\right]+d(a-1). \end{aligned}$$

(ii)

$$\begin{aligned}
n(a,d;k) &= \frac{1}{a}\sum_{\mathscr{C}} m_{\mathscr{C}} - \frac{a-1}{2} \\
&= \frac{1}{a}\sum_{y=1}^{a-1} a\left(1 + \left[\frac{y-1}{k-1}\right]\right) + dy - \frac{a-1}{2} \\
&= \sum_{y=0}^{a-2}\left(1 + \left[\frac{y}{k-1}\right]\right) + \frac{d(a-1)}{2} - \frac{a-1}{2} \\
&= \sum_{y=0}^{[(a-2)/(k-1)](k-1)-1}\left(1 + \left[\frac{y}{k-1}\right]\right) \\
&\quad + \sum_{y=[a-2/k-1](k-1)}^{a-2}\left(1 + \left[\frac{y}{k-1}\right]\right) + \frac{(a-1)(d-1)}{2} \\
&= (k-1)\left(1 + 2 + \cdots + \left[\frac{a-2}{k-1}\right]\right) \\
&\quad + (1+t)\left(1 + \left[\frac{a-2}{k-1}\right]\right) + \frac{(a-1)(d-1)}{2}, \\
&\quad \text{where } t \equiv a - 2 \pmod{(k-1)} \\
&= \frac{1}{2}\left(1 + \left[\frac{a-2}{k-1}\right]\right)\left((k-1)\left[\frac{a-2}{k-1}\right] + t + 2 + t\right) \\
&\quad + \frac{(a-1)(d-1)}{2} \\
&= \frac{1}{2}\left(1 + \left[\frac{a-2}{k-1}\right]\right)(a+t) + \frac{(a-1)(d-1)}{2}, \\
&\qquad \text{where } t \equiv a - 2 \pmod{(k-1)}. \quad \square
\end{aligned}$$

ACKNOWLEDGMENT. The author wishes to thank Preety Nigam and the referee for their helpful suggestions.

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