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On Sums of Positive Integers That Are Not of the Form $ax + by$

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If a and b are coprime and positive integers, then the set $a\mathbb{Z} + b\mathbb{Z} = \{ax + by : x, y \in \mathbb{Z}\} = \mathbb{Z}$. The subset $\Gamma(a, b) := \{ax + by : x, y \geq 0\}$ of $a\mathbb{Z} + b\mathbb{Z}$ is only slightly more difficult to describe. It is obvious that $\Gamma(a, b)$ is an infinite set, but not so obvious that $\Gamma^c(a, b) = \mathbb{N} \setminus \Gamma(a, b)$ is finite. Indeed, it was known to Sylvester [**2**] that the largest integer in $\Gamma^c(a, b)$ is $ab - a - b = (a - 1)(b - 1) - 1$. It is not hard to show this, nor even that the number of elements in $\Gamma^c(a, b)$ is $(a - 1)(b - 1)/2$. However, there appears to be no easy way to sum the integers in $\Gamma^c(a, b)$; for instance, a generating function is used to do this in [**1**]. The purpose of this note is to show that the sum $s(a, b)$ can be determined directly, and quite easily. In fact, we do a bit more.

Given k positive integers $a_1, a_2, \ldots, a_k$ with $\gcd(a_1, a_2, \ldots, a_k) = 1$, we consider the complement of the set $\Gamma(a_1, a_2, \ldots, a_k) := \{a_1x_1 + a_2x_2 + \cdots + a_kx_k : x_i \geq 0\}$ in $\mathbb{N}$. Observe that if $n \in \Gamma(a_1, a_2, \ldots, a_k)$, then $n + ma_1 \in \Gamma(a_1, a_2, \ldots, a_k)$ for every nonnegative integer m. As a consequence of the well-ordering of $\mathbb{N}$, there is a *least* positive integer in $\Gamma(a_1, a_2, \ldots, a_k)$ among those congruent to i modulo a_1 for each i with $1 \leq i \leq a_1 - 1$. We denote this minimum by m_i, and note that $\Gamma^c(a_1, a_2, \ldots, a_k)$ can be expressed as the union of $a_1 - 1$ arithmetic progressions, one for each i between 1 and $a_1 - 1$. The i th arithmetic progressions has first term i, last term $m_i - a_1$ and common difference a_1. This makes it easy to express the sum $s(a_1, a_2, \ldots, a_k)$ of the integers in $\Gamma^c(a_1, a_2, \ldots, a_k)$ in terms of these minima.

By the definition of the m_i's, $m_i - a_1$ is the *largest* positive integer in $\Gamma^c(a_1, a_2, \ldots, a_k)$ among those congruent to i modulo a_1. Hence the sum of elements in $\Gamma^c(a_1, a_2, \ldots, a_k)$ congruent to i modulo a_1 is easily seen to be $(m_i - i)(m_i + i - a_1)/2a_1$. It follows that

$$s(a_1, a_2, \dots, a_k) = \frac{1}{2a_1} \sum_{i=1}^{a_1-1} (m_i^2 - i^2) - \frac{1}{2} \sum_{i=1}^{a_1-1} (m_i - i) \tag{1}$$

$$= \frac{1}{2a_1} \sum_{i=1}^{a_1-1} m_i^2 - \frac{1}{2} \sum_{i=1}^{a_1-1} m_i - \frac{{a_1}^2 - 1}{12}. \tag{2}$$

We record this as a Lemma:

Lemma 1. *Let $a_1, a_2, \dots, a_k$ be positive coprime integers. For each i, $1 \le i \le a_1 - 1$, let m_i denote the least positive integer congruent to $i \mod a_1$ in $\Gamma(a_1, a_2, \dots, a_k) = \{a_1x_1 + a_2x_2 + \dots + a_kx_k : x_i \ge 0\}$. Then*

$$s(a_1, a_2, \dots, a_k) = \frac{1}{2a_1} \sum_{i=1}^{a_1-1} m_i^2 - \frac{1}{2} \sum_{i=1}^{a_1-1} m_i - \frac{{a_1}^2 - 1}{12}.$$

Theorem 1. *Let a, b be positive coprime integers. Then*

$$s(a, b) = \frac{1}{12}(a-1)(b-1)(2ab - a - b - 1).$$

Proof. This is a simple consequence of the fact that $\{m_i : 1 \le i \le a-1\} = \{bi : 1 \le i \le a-1\}$ and equation (1). We have

$$\begin{aligned} s(a, b) &= \frac{1}{2a}(b^2 - 1) \sum_{i=1}^{a-1} i^2 - \frac{1}{2}(b-1) \sum_{i=1}^{a-1} i \\ &= \frac{1}{12}(a-1)(b-1)\Big\{(2a-1)(b+1) - 3a\Big\} \\ &= \frac{1}{12}(a-1)(b-1)(2ab - a - b - 1). \qquad \blacksquare \end{aligned}$$

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