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ON THE DENSITY OF INTEGRAL SETS WITH MISSING DIFFERENCES

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Abstract

For a given set M of positive integers, a well-known problem of Motzkin asks for determining the maximal density $\mu(M)$ among sets of nonnegative integers in which no two elements differ by M. The problem is completely settled when $|M| \leq 2$, and some partial results are known for several families of M for $|M| \geq 3$. In this paper, we consider the case $M = \{a, b, c\}$, with c a multiple of a or b. In most cases, we obtain lower bounds for $\mu(M)$, which are conjecturally the exact values of $\mu(M)$, while in some we obtain the exact value of $\mu(M)$.

1. Introduction

For $x \in \mathbb{R}$ and a set S of nonnegative integers, let $S(x)$ denote the number of elements $n \in S$ such that $n \leq x$. The upper and lower densities of S, denoted by $\overline{\delta}(S)$ and $\underline{\delta}(S)$ respectively, are given by

$$\overline{\delta}(S) := \limsup_{x\to\infty} \frac{S(x)}{x}, \quad \underline{\delta}(S) := \liminf_{x\to\infty} \frac{S(x)}{x}.$$

If $\overline{\delta}(S) = \underline{\delta}(S)$, we denote the common value by $\delta(S)$, and say that S has density $\delta(S)$. Given a set of positive integers M, S is said to be an M-set if $a \in S$, $b \in S$ imply $a - b \notin M$. Motzkin in [3] asked to determine $\mu(M)$ given by

$$\mu(M) := \sup_S \overline{\delta}(S)$$

where S varies over the class of all M-sets. Cantor & Gordon in [1] showed the existence of $\mu(M)$ for any M, determined $\mu(M)$ when $|M| \leq 2$:

$$\begin{aligned} \mu(\{m_1\}) &= \frac{1}{2}, \\ \mu(\{m_1, m_2\}) &= \frac{\lfloor (m_1+m_2)/2 \rfloor}{m_1+m_2} \quad \text{for} \quad \gcd(m_1, m_2) = 1, \end{aligned}$$

and gave the following lower bound for $\mu(M)$:

$$\mu(M) \geq \sup_{\gcd(k,m)=1} \frac{1}{m} \min_i |km_i|_m,$$

where m_i are the elements of M and $|x|_m$ denotes the absolute value of the absolutely least remainder of x mod m. Haralambis in [2] gave the equivalent expressions for the right-hand side expression of the above inequality:

$$\begin{aligned} d_1(M) &= \sup_{x \in (0,1)} \min_i \|xm_i\|, \\ d_2(M) &= \sup_{\gcd(k,m)=1} \frac{1}{m} \min_i |km_i|_m, \\ d_3(M) &= \max_{\substack{m=m_j+m_\ell \\ 1 \leq k \leq \frac{m}{2}}} \frac{1}{m} \min_i |km_i|_m \end{aligned}$$

where $\|x\|$ denotes the distance from the nearest integer. Thus $d_1(M) = d_2(M) = d_3(M)$, and we denote this common value by $d(M)$. Hence $d(M)$ serves as a lower bound for $\mu(M)$. A useful upper bound for $\mu(M)$ is due to Haralambis in [2]:

$\mu(M) \leq \alpha$ provided there exists a positive integer k such that $S(k) \leq (k+1)\alpha$ for every M-set S with $0 \in S$.

In fact, Haralambis in [2] conjectured that $\mu(M) = d(M)$ for $|M| = 3$, so that, conjecturally, determining $d(M) = d_3(M)$ gives the value of $\mu(M)$.

We consider the problem for the families $M = \{a, b, c\}$, where c is a multiple of a or b. By a result of Cantor & Gordon in [1], we know that $\mu(kM) = \mu(M)$. Thus, it is no loss of generality to assume that $\gcd(a, b) = 1$, and that $a < b$. In most cases, we determine the value of $d(M)$, which is the lower bound for $\mu(M)$ and conjecturally equal to it, and in some cases we determine the value of $\mu(M)$.

2. Exact Results

We begin by dealing with one special case where we determine $\mu(M)$. We use the upper and lower bounds for $\mu(M)$ to achieve this.

Theorem 1. *Let $M = \{a, b, c\}$, where $a+b$ is odd, $\gcd(a, b) = 1$, $c \in \{na, nb\}$, and $n \equiv \pm 1 \pmod{a+b}$. Then*

$$\mu(M) = \frac{a+b-1}{2(a+b)}.$$

Proof. By Cantor & Gordon's result we have $\mu(M) \le \mu(\{a, b\}) = \frac{a+b-1}{2(a+b)}$. For the reverse inequality, choose x such that $ax \equiv \frac{a+b-1}{2} \pmod{a+b}$. Since $cx \equiv \pm ax \equiv \mp bx \pmod{a+b}$, $\mu(M) \ge \frac{a+b-1}{2(a+b)}$. Hence the result. □

3. The Case $M = \{a, b, nb\}$

In this section, we deal with the family $M = \{a, b, nb\}$, with $a < b$, $\gcd(a, b) = 1$, and $n \ge 2$. We compute $d_3(M)$ by comparing the rational numbers in the three cases, as mentioned in Section 1.

Theorem 2. *Let $M = \{a, b, nb\}$, where $a < b$, $\gcd(a, b) = 1$, and a, b are odd integers. Then*

$$\mu(M) = \begin{cases} \frac{n}{2(n+1)} & \text{if } n \text{ is even;} \\ \frac{1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By Cantor and Gordon's result we have $\mu(M) \le \mu(\{a, b\}) = \frac{1}{2}$. If n is odd, then $\{1, 3, 5, \ldots\}$ is an M-set. Hence $\mu(M) = \frac{1}{2}$ in this case. If n is even, then $\mu(M) \le \mu(\{b, nb\}) = \mu(\{1, n\}) = \frac{n}{2(n+1)}$. To show the reverse inequality, let $m = (n+1)b$. Observe that m is odd. Choose $x \equiv \frac{m-1}{2} \pmod{m}$. Then

$$ax \equiv \tfrac{m-a}{2} \pmod{m}, \quad -nbx \equiv bx \equiv \tfrac{m-b}{2} \pmod{m}.$$

Since $\frac{1}{2}(m-a) > \frac{1}{2}(m-b) = \frac{1}{2}nb$, we have $\mu(M) \ge d(M) \ge \frac{nb}{2m} = \frac{n}{2(n+1)}$. This completes the proof. □

Lemma 1. *For $r, s \ge 0$, let*

$$A_r := \{2r(a+b) + 2t - 1 : 1 \le t \le b\}, \quad B_s := \{2s(a+b) + 2b + 2t - 1 : 1 \le t \le a\}.$$

Then $\{A_0, A_1, \ldots, B_0, B_1, \ldots\}$ partitions the set of positive odd integers $2\mathbb{N} - 1$.

Proof. Observe that $|A_r| = b$ and $|B_s| = a$ for each $r, s \ge 0$, and that $A_{r+1} = A_r + 2(a+b)$ and $B_{s+1} = B_s + 2(a+b)$. The lemma now follows from the observation that $\{A_0, B_0\}$ partitions the odd integers in the interval $[1, 2(a+b) - 1]$. □

Theorem 3. *Let $M = \{a, b, nb\}$, where $a < b$, $\gcd(a, b) = 1$, $a + b$ and n are odd integers. Let the family of sets $\{A_r\}_{r\geq 0}$ and $\{B_s\}_{s\geq 0}$ be defined as in Lemma 1. If $n \not\equiv \pm 1 \pmod{a+b}$, then*

$$d(M) = \begin{cases} \frac{m-(2rb+2t-1)}{2m} & \text{if } n \in A_r \text{ and where } m = a + nb; \\ \frac{m-2(s+1)b}{2m} & \text{if } n \in B_s \text{ and where } m = (n+1)b. \end{cases}$$

Proof. We compute $d(M)$ by using the expression for $d_3(M)$ in Section 1. Thus there are three choices for m, and we determine $d_3(M)$ by comparing the three rational numbers corresponding to these case. By Lemma 1, n belongs to a unique set among the two families $\{A_r\}_{r\geq 0}$ and $\{B_s\}_{s\geq 0}$.

CASE I: $(m = a + nb)$ Observe that m is odd, and that $\gcd(b, m) = 1$.

Subcase (i): $(n \in A_r)$ Choose x such that

$$bx \equiv \tfrac{m-(2rb+1)}{2} \pmod{m}.$$

Thus $2ax \equiv -2nbx \equiv n(2rb+1) = 2r(m-a)+n \equiv n-2ra = 2rb+2t-1 \pmod{m}$, and

$$ax \equiv -\tfrac{m-(2rb+2t-1)}{2} \pmod{m}.$$

Since $nbx \equiv -ax \pmod{m}$,

$$\min\big\{|ax|_m, |bx|_m, |nbx|_m\big\} = \tfrac{m-(2rb+2t-1)}{2}. \tag{1}$$

We now show that

$$\min\big\{|ay|_m, |by|_m, |nby|_m\big\} \leq \tfrac{m-(2rb+2t-1)}{2} = \tfrac{m-1}{2} - \ell,$$

for each y, $1 \leq y \leq \frac{1}{2}(m-1)$, where $\ell = rb+t-1$. Let $\mathcal{I} := \big[\frac{m-1}{2} - \ell, \frac{m+1}{2} + \ell\big]$ and $\mathcal{J} := \big(\frac{m-1}{2} - \ell, \frac{m+1}{2} + \ell\big)$. We show that, for $1 \leq y \leq \frac{1}{2}(m-1)$, if $by \pmod{m} \in \mathcal{I}$, then $ay \pmod{m} \notin \mathcal{J}$. Accordingly, write

$$by \equiv \tfrac{m-1}{2} - \ell + i \pmod{m}.$$

Then $by \pmod{m} \in \mathcal{I}$ if and only if $0 \leq i \leq 2\ell+1$, and $2ay \equiv -2nby \equiv n\big(1+2(\ell-i)\big) \pmod{m}$. Since $n\big(1+2(\ell-i)\big)$ is odd, we get

$$ay \equiv \tfrac{m-1}{2} + \tfrac{n+1}{2} + n(\ell - i) \pmod{m}.$$

To show that $ay \pmod{m} \notin \mathcal{J}$, we consider the two cases $0 \leq i \leq \ell$ and $\ell + 1 \leq i \leq 2\ell + 1$.

First consider the case $0 \leq i \leq \ell$. For each k, $0 \leq k \leq r$, define

$$I_k := \Big[\ell - \tfrac{1}{n}\big((k+1)m - \ell - \tfrac{n+1}{2}\big), \ell - \tfrac{1}{n}\big(km - \ell - \tfrac{n+1}{2}\big) - 1\Big], \quad J := \{\ell - kb : 1 \leq k \leq r\} \cup \{\ell\}.$$

Then it can be shown that $I_0 \cup I_1 \cup \ldots \cup I_r \cup J$ contains the set $\{0, 1, 2, \ldots, \ell\}$. A simple computation shows that

$$ay \in \begin{cases} \left(\frac{m+1}{2} + \ell, m\right) & \text{if } i = \ell; \\ \left[0, \frac{m-1}{2} - \ell\right) & \text{if } i \in J, i \neq \ell; \\ \left[km + \frac{m+1}{2} + \ell, (k+1)m + \frac{m-1}{2} - \ell\right] & \text{if } i \in I_k \text{ with } 0 \leq k \leq r. \end{cases}$$

For the cases $\ell + 1 \leq i \leq 2\ell + 1$, define for $0 \leq k \leq r$,

$$I'_k := \left[\ell + \tfrac{1}{n}\left(km - \ell + \tfrac{n-1}{2}\right) + 1, \ell + \tfrac{1}{n}\left((k+1)m - \ell + \tfrac{n-1}{2}\right)\right], \quad J' := \{\ell{+}1{+}kb : 1 \leq k \leq r\}.$$

Then it can be shown that $I'_0 \cup I'_1 \cup \ldots \cup I'_r \cup J'$ contains the set $\{\ell{+}1, \ell{+}2, \ldots, 2\ell{+}1\}$, and that $ay \pmod m \notin \mathcal{J}$. This completes the subcase when $n \in A_r$.

Subcase (ii): ($n \in B_s$) Choose x such that

$$bx \equiv \frac{m - \big(2(s+1)b+1\big)}{2} \pmod m.$$

The computation in subcase (i) can be employed to show that

$$ax \equiv -\frac{m - \big(2(s+1)b - 2(a-t) - 1\big)}{2} \pmod m,$$

so that

$$\min\left\{|ax|_m, |bx|_m, |nbx|_m\right\} = \frac{m - \big(2(s+1)b+1\big)}{2}. \tag{2}$$

To show that

$$\min\left\{|ay|_m, |by|_m, |nby|_m\right\} \leq \frac{m - \big(2(s+1)b+1\big)}{2} = \frac{m-1}{2} - \ell,$$

for each y, $1 \leq y \leq \frac{1}{2}(m-1)$, where $\ell = (s+1)b$, we mimic the proof in subcase (i), the only change being in the value of ℓ. All notations and congruences carry through, so the derivations are omitted. This completes the subcase when $n \in B_s$, and Case I.

Case II: **(**$m = (n+1)b$**)** Observe that m is even. As in Case I, we consider two subcases.

Subcase (i): ($n \in A_r$) For each x, $1 \leq x \leq \frac{1}{2}m$, we show that

$$\min\{|ax|_m, |bx|_m, |nbx|_m\} \leq \frac{m}{2} - (rb + t). \tag{3}$$

Suppose $|bx|_m > \frac{m}{2} - (rb + t)$. Then

$$\lambda m + \frac{m}{2} - rb - t < bx < \lambda m + \frac{m}{2} + rb + t$$

for some integer λ. Thus $x \in \left[\lambda(n+1) + \frac{n+1}{2} - r, \lambda(n+1) + \frac{n+1}{2} + r\right]$. Write $x \equiv \frac{n+1}{2} - r + i \pmod{n+1}$, with $0 \le i \le 2r$. Since $\frac{n+1}{2} = r(a+b) + t$, it is easy to verify that

$$ax \equiv -ar + ai \equiv \tfrac{n+1}{2} + br + t + ai \pmod{n+1} \text{ when } b \text{ is odd}$$

and

$$ax \equiv br + t + ai \pmod{n+1} \text{ when } b \text{ is even.}$$

Now $2ar = (n+1) - 2(br+t)$. Hence $0 \le ai \le (n+1) - 2(rb+t)$ for $0 \le i \le 2r$, and $|ax|_m \le \frac{m}{2} - (br+t)$, as desired. This completes the subcase when $n \in A_r$.

Subcase (ii): ($n \in B_s$) As in subcase (i), we can choose an integer λ such that with $x = \lambda(n+1) + \frac{n+1}{2} - (s+1)$, we have

$$bx \equiv \tfrac{m}{2} - (s+1)b \pmod{m}, \quad ax \equiv \tfrac{m}{2} + (s+1)b + (t-a) \pmod{m}.$$

Hence

$$\min\{|ax|_m, |bx|_m, |nbx|_m\} = \tfrac{m}{2} - (s+1)b. \tag{4}$$

Again, an argument similar to the one in subcase (i) shows that

$$\min\{|ay|_m, |by|_m, |nby|_m\} \le \tfrac{m}{2} - (s+1)b$$

for each y, $1 \le y \le \frac{1}{2}m$. This completes the argument in Case II.

Case III: ($m = a+b$) Observe that m is odd, and that $\gcd(a,m) = 1 = \gcd(b,m)$. Choose x such that $ax \equiv -bx \equiv -\frac{a+b-1}{2} \pmod{m}$. Since n is odd, it is easy to see that

$$nbx \equiv \tfrac{m-n}{2} \equiv \pm\tfrac{m-1}{2} \pmod{m}$$

if and only if $n \equiv \pm 1 \pmod{2m}$. Since $\frac{m-1}{2}$ is the maximum absolute remainder mod m and since $n \equiv \pm 1 \pmod{2(a+b)}$ is excluded by assumption,

$$\min\{|ax|_m, |bx|_m, |nbx|_m\} \le \tfrac{a+b-3}{2}$$

for each x, $1 \le x \le \frac{1}{2}(m-1)$.

To determine $d(M)$, we consider the two cases $n \in A_r$ and $n \in B_s$ separately, and compare the values given by the three cases. For $n \in A_r$, $n \not\equiv \pm 1 \pmod{2(a+b)}$, observe that

$$\tfrac{(n+1)b - 2(rb+t)}{2(n+1)b} = \tfrac{1}{2} - \tfrac{rb+t}{(n+1)b} < \tfrac{1}{2} - \tfrac{2(rb+t)-1}{2(a+nb)} = \tfrac{(a+nb)-(2rb+2t-1)}{2(a+nb)},$$

and

$$\tfrac{a+b-3}{2(a+b)} = \tfrac{1}{2} - \tfrac{3}{2(a+b)} < \tfrac{1}{2} - \tfrac{2(rb+t)-1}{2(a+nb)} = \tfrac{(a+nb)-(2rb+2t-1)}{2(a+nb)}.$$

Thus the upper bounds for $d(M)$ in Cases II and III are each less than the value of $d(M)$ in Case I. Hence, in this case

$$d(M) = \tfrac{1}{2} - \tfrac{2(rb+t)-1}{2(a+nb)} = \tfrac{(a+nb)-(2rb+2t-1)}{2(a+nb)}.$$

For $n \in B_s$, $n \not\equiv \pm 1 \,(\mathrm{mod}\ 2(a+b))$, we have

$$\tfrac{(a+nb)-(2(s+1)b+1)}{2(a+nb)} = \tfrac{1}{2} - \tfrac{(s+1)b+\frac{1}{2}}{a+nb} < \tfrac{1}{2} - \tfrac{s+1}{n+1} = \tfrac{(n+1)-2(s+1)}{2(n+1)},$$

and

$$\tfrac{a+b-3}{2(a+b)} = \tfrac{1}{2} - \tfrac{3}{2(a+b)} < \tfrac{1}{2} - \tfrac{s+1}{n+1} = \tfrac{(n+1)-2(s+1)}{2(n+1)}.$$

Thus the upper bounds for $d(M)$ in Cases I and III are each less than the value of $d(M)$ in Case II, and

$$d(m) = \tfrac{1}{2} - \tfrac{s+1}{n+1} = \tfrac{(n+1)-2(s+1)}{2(n+1)}$$

in this case. This completes the comparison, and the proof of the theorem. □

Lemma 2. *For $r, s \geq 0$, let*

$$A'_r := \{(2r+1)(a+b)+2t-1 : 1 \leq t \leq b\}, \quad B'_s := \{(2s-1)(a+b)+2b+2t-1 : 1 \leq t \leq a\}.$$

Then $\{A'_0, A'_1, \ldots, B'_0, B'_1, \ldots\}$ partitions the set $b - a + (2\mathbb{N} - 1)$.

Proof. Observe that $A'_r = A_r + (a+b)$ and $B'_s = B_s - (a+b)$ for each $r, s \geq 0$. The proof is similar to that of Lemma 1. We have $|A'_r| = b$, $|B'_s| = a$ for each $r, s \geq 0$, $A'_{r+1} = A'_r + 2(a+b)$ and $B'_{s+1} = B'_s + 2(a+b)$. The lemma now follows from the observation that $\{A'_0, B'_0\}$ partitions the even integers in the interval $[b-a+1, 3b+a-1]$. □

Theorem 4. *Let $M = \{a, b, nb\}$ where $a < b$, $\gcd(a,b) = 1$, $a+b$ is odd, $n \geq b-a+1$ and even. Let the family of sets $\{A'_r\}_{r\geq 0}$ and $\{B'_s\}_{s\geq 0}$ be as defined in Lemma 2. If $n \not\equiv \pm 1 \,(\mathrm{mod}\ 2(a+b))$, then*

$$d(M) = \begin{cases} \frac{m-\{(2r+1)b+2t-1\}}{2m} & \text{where } m = a+nb \text{ and } n \in A'_r; \\ \frac{m-(2s+1)b}{2m} & \text{where } m = (n+1)b \text{ and } n \in B'_s. \end{cases}$$

Proof. We use the method of proof given in Theorem 3, and place every even integer $n \geq b-a+1$ in a unique set among the two families $\{A'_r\}_{r\geq 0}$ and $\{B'_s\}_{s\geq 0}$.

Case I: **($m = a + nb$)** Observe that $\gcd(b, m) = 1$.

Subcase (i): ($n \in A'_r$) Choose x such that

$$bx \equiv \tfrac{m-[(2r+1)b+1]}{2} \pmod{m}.$$

This is an analogue of the corresponding subcase in Theorem 3 with $2r+1$ replacing $2r$. The argument of this subcase carries through if we make this replacement throughout this subcase. We omit the details. This completes the subcase when $n \in A'_r$.

Subcase (ii): ($n \in B'_s$) Choose x such that

$$bx \equiv \tfrac{m-[(2s+1)b+1]}{2} \pmod{m}.$$

This is an analogue of the corresponding subcase in Theorem 3 with $2s+1$ replacing $2(s+1)$. The argument of this subcase carries through if we make this replacement throughout this subcase. We omit the details. This completes the subcase when $n \in B'_s$, and Case I.

Case II: ($m = a + nb$)

Subcase (i): ($n \in A'_r$) This is an analogue of the corresponding subcase in Theorem 3 with $2r+1$ replacing $2r$, obtaining only an upper bound. We omit the details.

Subcase (ii): ($n \in B'_s$) This is an analogue of the corresponding subcase in Theorem 3 with $2s+1$ replacing $2(s+1)$. We again omit the details.

Case III: ($m = a + b$)

We may use the exact same computation of Case III in Theorem 3 for this case as well. The rest of the proof is the same as that given in Theorem 3, and is omitted. □

Theorem 5. *Let $M = \{a, b, nb\}$, where $a < b$, $\gcd(a,b) = 1$, $a+b$ is odd, $n \leq b-a-1$, and n is even. Then $\mu(M) = \frac{n}{2(n+1)}$.*

Proof. By Cantor and Gordon's result we have $\mu(M) \leq \mu(\{b, nb\}) = \mu(\{1, n\}) = \frac{n}{2(n+1)}$. For the reverse inequality, let $m = (n+1)b$ and choose x such that $x \equiv \frac{n}{2} \pmod{n+1}$. Then $x = \lambda(n+1) + \frac{n}{2}$ for some integer λ, and a simple calculation shows that

$$ax \equiv \tfrac{m-a-(n+1)}{2} \pmod{m} \iff (2\lambda+1)a \equiv -1 \pmod{b}.$$

If b is even, then a must be odd, and so any solution of $ay \equiv \pm 1 \pmod{b}$ is necessarily odd. If b is odd, then a must be even, and we may choose y to be odd by replacing y by $b-y$, if necessary. In either case, we may choose λ such that $a(2\lambda+1) \equiv \pm 1 \pmod{b}$, and hence satisfy $ax \equiv \frac{m-a-(n+1)}{2} \pmod{m}$. Since $bx \equiv \frac{nb}{2} = \frac{m-b}{2} \pmod{m}$ and $b \geq a+n+1$, it follows that $\mu(M) \geq \frac{nb}{2(n+1)b} = \frac{n}{2(n+1)}$. This completes the proof. □

4. The Case $M = \{a, b, na\}$

We deal with the family $M = \{a, b, nb\}$, with $a < b$, $\gcd(a, b) = 1$, and $n \geq 2$. The results are analogous to those obtained in Section 3, and proofs similar. We begin by considering the case where a, b are both odd. However, unlike the analogous case in Theorem 2, we are able to determine $d(M)$ only for all sufficiently large n.

Theorem 6. *Let $M = \{a, b, na\}$, where $a < b$, $\gcd(a, b) = 1$, a, b are odd integers, and $n \geq \frac{b(a+b-2)}{2a}$ and even. Then $d(M) = \frac{na}{2(na+b)}$.*

Proof. We compute $d(M)$ by using the expression for $d_3(M)$ in Section 1.

Case I: $(m = na + b)$ Observe that m is odd. Choose x such that $x \equiv \frac{m-1}{2} \pmod m$. Then

$$ax \equiv \tfrac{m-a}{2} \pmod m \quad bx \equiv \tfrac{m-b}{2} \pmod m.$$

Thus

$$\min\{|ax|_m, |bx|_m, |nax|_m\} = \tfrac{m-b}{2} = \tfrac{na}{2}. \tag{5}$$

We now show that

$$\min\{|ay|_m, |by|_m, |nay|_m\} \leq \tfrac{m-b}{2}$$

for each y, $1 \leq y \leq \frac{1}{2}(m-1)$, by an argument similar to the one in Theorem 2. Let $\mathcal{I} := \left[\frac{m-b}{2}, \frac{m+b}{2}\right]$. We show that, for $1 \leq y \leq \frac{1}{2}(m-1)$, $by \pmod m \in \mathcal{I}$ and $ay \pmod m \in \mathcal{I}$ only when $y \equiv \frac{m-1}{2} \pmod m$. With $y \equiv \frac{m-1}{2} + i \pmod m$, a simple calculation shows that

$$by \pmod m \in \mathcal{I} \Longleftrightarrow i \in \left[k\tfrac{m}{b}, k\tfrac{m}{b} + 1\right]$$

for some integer k, with $0 \leq k \leq b-1$.

If $k = 0$, $i = 0$ gives $y \equiv \frac{m-1}{2} \equiv x \pmod m$ while $i = 1$ gives $y \equiv -\frac{m-1}{2} \equiv -x \pmod m$. For $1 \leq k \leq b-1$, let $ka = qb + r$, where $0 \leq r \leq b-1$. In fact, $r \neq 0$ since $b \mid ka$ otherwise, and this is impossible since $\gcd(a, b) = 1$ and $1 \leq k \leq b-1$. A routine calculation shows that $ay \pmod m$ lies between $\frac{m-a}{2} + \frac{mr}{b} \pmod m$ and $\frac{m+a}{2} + \frac{mr}{b} \pmod m$, and another shows

$$\tfrac{m+b}{2} \leq \tfrac{m-a}{2} + \tfrac{mr}{b} < \tfrac{m+a}{2} + \tfrac{mr}{b} \leq m + \tfrac{m-b}{2},$$

the first and third inequalities being valid since $2na \geq b(a+b-2)$. This proves our claim that $ay \pmod m \notin \mathcal{I}$ for $1 \leq y < \frac{m-1}{2}$, and completes the proof in this case.

Case II: $(m = (n+1)a)$ As in Case I, m is odd. The same choice of x gives

$$\min\{|ax|_m, |bx|_m, |nax|_m\} = \tfrac{m-b}{2} = \tfrac{(n+1)a-b}{2}. \tag{6}$$

The proof of
$$\min\{|ay|_m, |by|_m, |nay|_m\} \le \tfrac{m-b}{2}$$
for each y, $1 \le y \le \frac{1}{2}(m-1)$ is similar to the one in Case I, and omitted.

Case III: **(**$m = a + b$**)** In this case m is even, and $\gcd(a,m) = \gcd(b,m) = 1$. Observe that
$$ax \equiv -bx \equiv \tfrac{m}{2} \pmod{m}$$
implies $nax \equiv 0 \,(\mathrm{mod}\ m)$ since n is even. Therefore
$$\min\{|ax|_m, |bx|_m, |nax|_m\} \le \tfrac{m}{2} - 1$$
for each x, $1 \le x \le \frac{1}{2}m$.

It is easy to check that $\frac{na}{2(b+na)} > \frac{(n+1)a-b}{2(n+1)a}$, and that $\frac{na}{2(b+na)} \ge \frac{a+b-2}{2(a+b)}$ if and only if $n \ge \frac{b(a+b-2)}{2a}$. Hence the result. □

Lemma 3. *For $r, s \ge 0$, let*
$$C_r := \{2r(a+b)+2t-1 : 1 \le t \le a\}, \quad D_s := \{2s(a+b)+2a+2t-1 : 1 \le t \le b\}.$$
Then $\{C_1, C_2, \ldots, D_1, D_2, \ldots\}$ partitions the set $2(a+b) + (2\mathbb{N} - 1)$.

Proof. Observe that the families $\{C_r\}_{r\ge 0}$ and $\{D_s\}_{s\ge 0}$ are obtained from the families $\{A_r\}_{r\ge 0}$ and $\{B_s\}_{s\ge 0}$ by interchanging a and b. Observe also that $|C_r| = a$ and $|D_s| = b$ for each $r, s \ge 0$, and that $C_{r+1} = C_r + 2(a+b)$ and $D_{s+1} = D_s + 2(a+b)$. The lemma now follows from the observation that $\{C_1, D_1\}$ partitions the odd integers in the interval $[2(a+b)+1, 4(a+b)-1]$. □

Theorem 7. *Let $M = \{a, b, na\}$ where $a < b$, $\gcd(a,b) = 1$, $a + b$ is odd, $n \ge 2(a+b)+1$ and odd. Let the family of sets $\{C_r\}_{r\ge 1}$ and $\{D_s\}_{s\ge 1}$ be defined as in Lemma 3. If $n \not\equiv \pm 1 \,(\mathrm{mod}\ a+b)$, then*
$$d(M) = \begin{cases} \frac{m-(2ra+2t-1)}{2m} & \text{if } n \in C_r \text{ and where } m = na+b; \\ \frac{m-2(s+1)a}{2m} & \text{if } n \in B_s,\ b \le 2(s+1)a + t, \text{ and where } m = (n+1)a. \end{cases}$$

Proof. The argument in Theorem 3 carries over with the roles of a and b interchanged. We omit the proof. □

Remark 1. *We remark that just as the families $\{C_r\}_{r\ge 1}$ and $\{D_s\}_{s\ge 1}$ are obtained from the families $\{A_r\}_{r\ge 0}$ and $\{B_s\}_{s\ge 0}$ by interchanging a and b, so are the corresponding results from Theorem 3. However, these formulae do not hold for $n \in C_0 \cup D_0$.*

Lemma 4. *For $r, s \geq 0$, let*

$$C'_r := \{(2r{+}1)(a{+}b){+}2t{-}1 : 1 \leq t \leq a\}, \quad D'_s := \{(2s{+}1)(a{+}b){+}2a{+}2t{-}1 : 1 \leq t \leq b\}.$$

Then $\{C'_0, C'_1, \ldots, D'_0, D'_1, \ldots\}$ partitions the set $(a+b)+(2\mathbb{N}-1)$.

Proof. Observe that $C'_r = C_r + (a+b)$ and $D'_s = D_s + (a+b)$ for each $r, s \geq 0$. The lemma now follows from Lemma 3. □

Theorem 8. *Let $M = \{a, b, na\}$ where $a < b$, $\gcd(a, b) = 1$, $a{+}b$ is odd, $n \geq a{+}b{+}1$ and even. Let the family of sets $\{C'_r\}_{r\geq 0}$ and $\{D'_s\}_{s\geq 0}$ be defined as in Lemma 4. If $n \not\equiv \pm 1 \pmod{a+b}$, then*

$$d(M) = \begin{cases} \frac{m-\big((2r+1)a+2t-1\big)}{2m} & \text{if } n \in C'_r \text{ and where } m = na+b; \\ \frac{m-(2s+3)a}{2m} & \text{if } n \in D'_s,\ b \leq (2s+3)a+t, \text{ and where } m = (n+1)a. \end{cases}$$

Proof. The argument in Theorem 4 carries over with the roles of a and b interchanged, and in addition, replacing s by $s+1$ in the case $n \in B'_s$. We omit the proof. □

5. Concluding Remarks

In the previous sections, we have been able to obtain exact values of $d(M)$ in many cases, and in some special cases, even the value of $\mu(M)$. We expect that the values of $d(M)$ are in fact equal to $\mu(M)$, although we have not been able to show this. There are, however, a few cases where the exact value of $d(M)$ has eluded us. We state this as the following concluding remark.

Remark 2. *Let $M = \{a, b, na\}$, with $a < b$ and $\gcd(a, b) = 1$. We have been unable to determine $d(M)$ in the following cases:*

(i) *$a+b$ is odd, $n \leq a+b-1$ and even, and satisfies $b \geq (2s+3)a+t$;*

(ii) *$a+b$ is odd, $n \leq 2(a+b)-1$ and odd, and satisfies $b \geq 2(s+1)a+t$;*

(iii) *a, b odd, $n < \frac{b(a+b-2)}{2a}$ and even.*

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