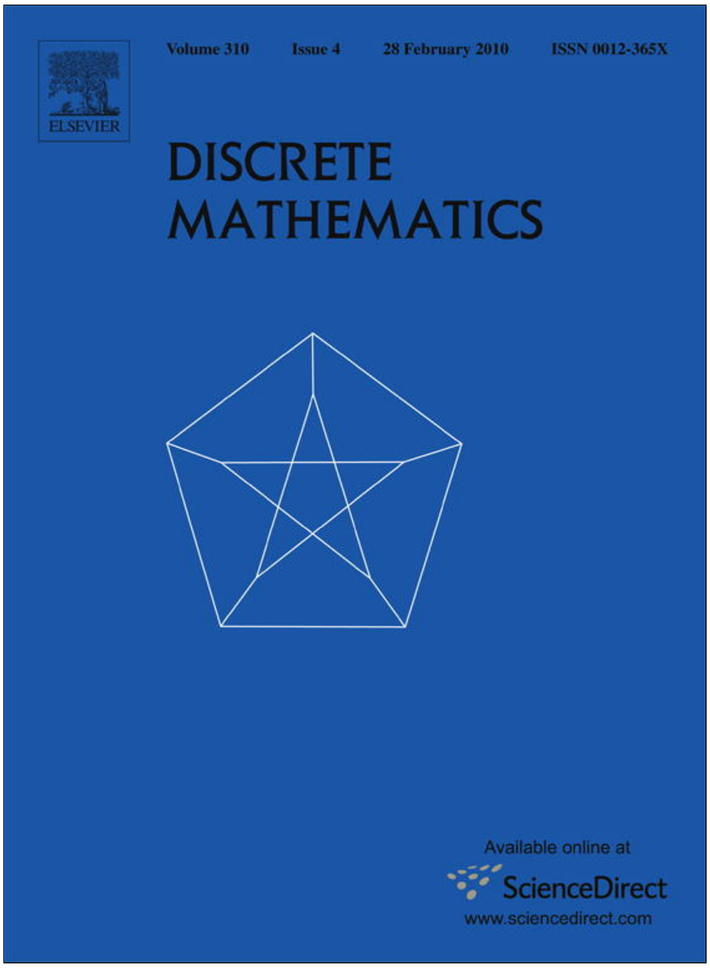
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A short constructive proof of the Erdős–Gallai characterization of graphic lists

Amitabha Tripathi [a,*], Sushmita Venugopalan [b,1], Douglas B. West [c]

[a] *Indian Institute of Technology, Hauz Khas, New Delhi, India*
[b] *Rutgers University, Piscataway, NJ 08854, United States*
[c] *University of Illinois, Urbana, IL 61801, United States*

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ABSTRACT

Erdős and Gallai proved that a nonincreasing list $(d_1, \ldots, d_n)$ of nonnegative integers is the list of degrees of a graph (with no loops or multi-edges) if and only if the sum is even and the list satisfies $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$ for $1 \leq k \leq n$. We give a short constructive proof of the characterization.

A list of nonnegative integers is *graphic* if it is the list of vertex degrees of a graph, where our model of *graph* forbids loops and repeated edges. Historically, such lists have also been called *graphic sequences*. A graph with degree list d is a *realization* of d.

Many characterizations of graphic lists are known; Sierksma and Hoogeveen [10] list seven criteria involving inequalities on the list elements. With additional results, these also appear in [9]. Additional characterizations are due to Havel [7] and Hakimi [5], Koren [8], and probably others.

The best-known explicit characterization is that by Erdős and Gallai [4]. Many proofs of it have been given, including that by Berge [2] (using network flow or Tutte's f-Factor Theorem), Harary [6] (a lengthy induction), Choudum [3], Aigner–Triesch [1] (using ideals in the dominance order), Tripathi–Tyagi [11] (indirect proof), etc. The purpose of this note is to give a short direct proof that constructs a graph whose degree list is the given list.

Theorem 1 (*Erdős–Gallai [4]*). *A list $(d_1, \ldots, d_n)$ of nonnegative integers in nonincreasing order is graphic if and only if its sum is even and, for each integer k with $1 \leq k \leq n$,*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}, \quad \text{for } 1 \leq k \leq n. \tag{1}$$

Proof. Necessity is immediate and standard: each edge is counted twice to yield an even sum, and the right side is the maximum contribution to the sum of the first k degrees from edges induced by the corresponding vertices and edges to the remaining vertices.

* Corresponding author.
E-mail addresses: atripath@maths.iitd.ac.in (A. Tripathi), sushvenu@math.rutgers.edu (S. Venugopalan), west@math.uiuc.edu (D.B. West).

1 Work done while at Indian Institute of Technology, New Delhi, India.

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For sufficiency, let a *subrealization* of a nonincreasing list $(d_1, \ldots, d_n)$ be a graph with vertices $v_1, \ldots, v_n$ such that $d(v_i) \leq d_i$ for $1 \leq i \leq n$, where $d(v_i)$ denotes the degree of v_i. Given a list $(d_1, \ldots, d_n)$ with an even sum that satisfies (1), we construct a realization through successive subrealizations. The initial subrealization has n vertices and no edges.

In a subrealization, the *critical index* r is the largest index such that $d(v_i) = d_i$ for $1 \leq i < r$. Initially, $r = 1$ unless the list is all 0, in which case the process is complete. While $r \leq n$, we obtain a new subrealization with smaller deficiency $d_r - d(v_r)$ at vertex v_r while not changing the degree of any vertex v_i with $i < r$ (the degree list increases lexicographically). The process can only stop when the subrealization is a realization of d.

Let $S = \{v_{r+1}, \ldots, v_n\}$. We maintain the condition that S is an independent set, which certainly holds initially. Write $v_i \leftrightarrow v_j$ when $v_iv_j \in E(G)$; otherwise, $v_i \not\leftrightarrow v_j$.

Case (0) *$v_r \not\leftrightarrow v_i$ for some vertex v_i such that $d(v_i) < d_i$*. Add the edge v_rv_i.

Case (1) *$v_r \not\leftrightarrow v_i$ for some i with $i < r$*. Since $d(v_i) = d_i \geq d_r > d(v_r)$, there exists $u \in N(v_i) - (N(v_r) \cup \{v_r\})$, where $N(z) = \{y\colon z \leftrightarrow y\}$. If $d_r - d(v_r) \geq 2$, then replace uv_i with $\{uv_r, v_iv_r\}$. If $d_r - d(v_r) = 1$, then since $\sum d_i - \sum d(v_i)$ is even there is an index k with $k > r$ such that $d(v_k) < d_k$. Case 0 applies unless $v_r \leftrightarrow v_k$; replace $\{v_rv_k, uv_i\}$ with $\{uv_r, v_iv_r\}$.

Case (2) *$v_1, \ldots, v_{r-1} \in N(v_r)$, and $d(v_k) \neq \min\{r, d_k\}$ for some k with $k > r$*. In a subrealization, $d(v_k) \leq d_k$. Since S is independent, $d(v_k) \leq r$. Hence $d(v_k) < \min\{r, d_k\}$, and Case 0 applies unless $v_k \leftrightarrow v_r$. Since $d(v_k) < r$, there exists i with $i < r$ such that $v_k \not\leftrightarrow v_i$. Since $d(v_i) > d(v_r)$, there exists $u \in N(v_i) - (N(v_r) \cup \{v_r\})$. Replace uv_i with $\{uv_r, v_iv_k\}$.

Case (3) *$v_1, \ldots, v_{r-1} \in N(v_r)$, and $v_i \not\leftrightarrow v_j$ for some i and j with $i < j < r$*. Case 1 applies unless $v_i, v_j \in N(v_r)$. Since $d(v_i) \geq d(v_j) > d(v_r)$, there exist $u \in N(v_i) - (N(v_r) \cup \{v_r\})$ and $w \in N(v_j) - (N(v_r) \cup \{v_r\})$ (possibly $u = w$). Since $u, w \notin N(v_r)$, Case 1 applies unless $u, w \in S$. Replace $\{uv_i, wv_j\}$ with $\{v_iv_j, uv_r\}$.

If none of these Cases applies, then $v_1, \ldots, v_r$ are pairwise adjacent, and $d(v_k) = \min\{r, d_k\}$ for $k > r$. Since S is independent, $\sum_{i=1}^{r} d(v_i) = r(r-1) + \sum_{k=r+1}^{n} \min\{r, d_k\}$. By (1), $\sum_{i=1}^{r} d_i$ is bounded by the right side. Hence we have already eliminated the deficiency at vertex r. Increase r by 1 and continue. □

The proof can be implemented as an algorithm to construct a realization of the degree list. Since the subrealization improves lexicographically with each step, the number of steps is at most $\sum d_i$. To bound the time for each step, we maintain the graph as lists of neighbors and non-neighbors for each vertex. We look through the non-neighbors of v_r to see if Case 0 or Case 1 applies. To apply Case 1 we access lists twice to find u and possibly check degrees of the high-indexed vertices to find k. The implementation of Cases 2 and 3 involve similar operations. Each step is implemented using a constant number of set-membership queries. Thus the running time is naively at most $O(n \sum d_i)$. With clever maintenance of sets using sophisticated data structures involving trees, the factor of n can be reduced.

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