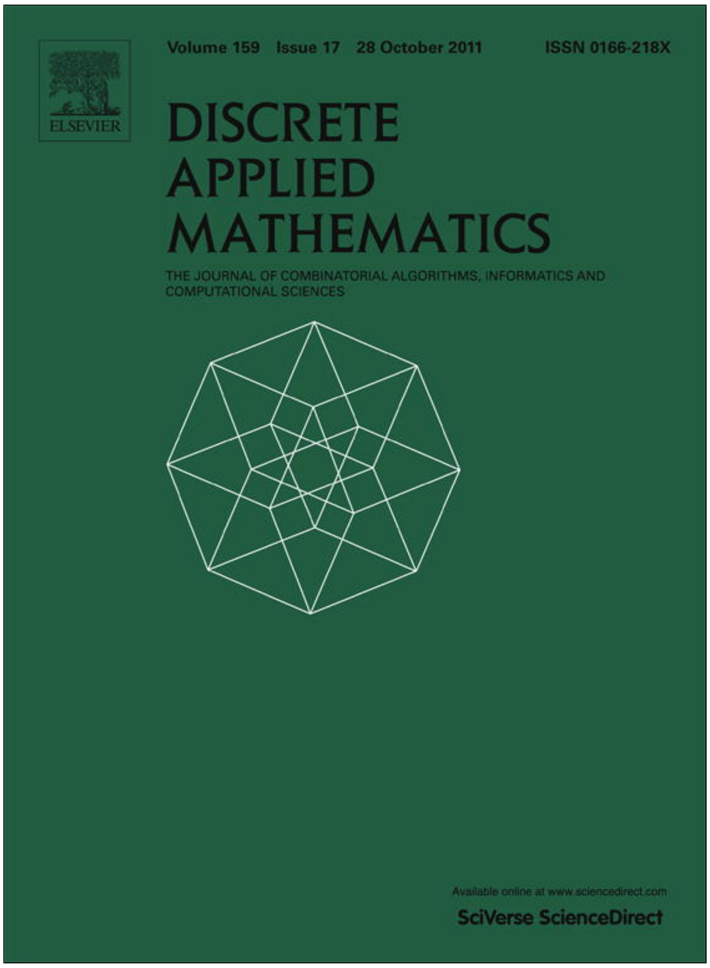
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Note

Constructive extensions of two results on graphic sequences

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ABSTRACT

A list of nonnegative integers is *graphic* if it is the list of vertex degrees of a graph. Erdős–Gallai characterized graphic lists, and Gale and Ryser, independently, provided one for a bipartite graph, given two lists of nonnegative integers. We give a constructive proof of an extension of these two results.

1. Introduction

A list of nonnegative integers is *graphic* if it is the list of vertex degrees of a graph, where we assume that a graph is simple. Such lists are called *graphic sequences*. A graph with degree list d is a *realization* of d. Many characterizations of graphic lists are known, of which one of the best-known explicit characterizations is that by Erdős and Gallai [2]. There have been several proofs of this characterization, including a short constructive proof in [7], wherein several other references are included.

Theorem EG (*Erdős–Gallai, [2]*). *A list $a_1, \ldots, a_n$ of nonnegative integers in nonincreasing order is graphic if and only if its sum is even and, for each integer k with $1 \leq k \leq n$,*

$$\sum_{i=1}^{k} a_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, a_i\}.$$

Significantly, the number of inequalities that need to be verified can be reduced to verifying only the number of *distinct* elements in the list; see [8]. Cai et al. [1], in answering a question of Niessen [5], obtained a generalization of the Erdős–Gallai result by allowing each member of the list of nonnegative integers to be chosen from a prescribed set of consecutive integers. Using Lovász's (g, f)-factor theorem [4], they show that only a slight modification in the inequality of Erdős–Gallai is the required characterization. We provide a direct, constructive proof of this characterization in Section 2.

The notion of graphic sequences can be extended to *bigraphic sequences*. These are two lists of nonnegative integers such that each list is the list of vertex degrees of a partite set of a bipartite graph. Gale [3], and Ryser [6], independently, gave a characterization of two lists that realize a bigraph using results on network flows.

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Theorem GR (*Gale [3], Ryser [6]*)**.** *Let* $\mathcal{P} := p_1, \ldots, p_m$ *and* $\mathcal{Q} := q_1, \ldots, q_n$ *be two lists of nonnegative integers with* $p_1 \geq \cdots \geq p_m$, $q_1 \geq \cdots \geq q_n$, *and* $\sum_{i=1}^m p_i = \sum_{j=1}^n q_j$, *then* $(\mathcal{P}, \mathcal{Q})$ *is bigraphic if and only if*

$$\sum_{j=1}^{k} q_j \leq \sum_{i=1}^{m} \min\{p_i, k\}$$

for $1 \leq k \leq n$.

Analogous to the case of graphic sequences, we consider two lists of intervals of nonnegative integers, and ask if there exists a bipartite graph such that the degrees of vertices of each partite set lies within the corresponding list. We characterize such pairs of lists that realize a bipartite graph, and prove this characterization by a constructive proof in Section 3.

2. A constructive extension of the Erdős–Gallai Theorem

Niessen [5] asked for the extension of result of Erdős–Gallai to characterizing lists of intervals $[a_1, b_1], \ldots, [a_n, b_n]$ such that there exists a simple graph with vertices $v_1, \ldots, v_n$ satisfying $a_i \leq d(v_i) \leq b_i$, $1 \leq i \leq n$. Cai et al. [1] provided such a characterization.

Definition 1. Let $\preceq$ denote the lexicographic ordering on $\mathbb{Z} \times \mathbb{Z}$:

$$(a, b) \preceq (a', b') \iff [(a < a') \vee \big((a = a')\&(b \leq b')\big)].$$

Let $\mathcal{A} := a_1, \ldots, a_n$ and $\mathcal{B} := b_1, \ldots, b_n$ be two lists of nonnegative integers. We say that the lists $\mathcal{A}, \mathcal{B}$ are in good order if

$$(a_{i+1}, b_{i+1}) \preceq (a_i, b_i)$$

for $1 \leq i \leq n - 1$.

For $0 \leq k \leq n$, define

$$I_k := \{i : \min\{i, b_i\} \geq k + 1\}$$

and

$$\epsilon(k) = \begin{cases} 1 & \text{if } a_i = b_i \text{ for } i \in I_k \text{ and } \sum_{i \in I_k} b_i + k|I_k| \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem CDZ (*Cai et al. [1]*)**.** *Let* $\mathcal{A} := a_1, \ldots, a_n$ *and* $\mathcal{B} := b_1, \ldots, b_n$ *be two lists of nonnegative integers in good order such that* $a_i \leq b_i$ *for* $1 \leq i \leq n$. *Then there exists a simple graph G with vertices* $v_1, \ldots, v_n$ *such that* $a_i \leq d(v_i) \leq b_i$ *for* $1 \leq i \leq n$ *if and only if*

$$\sum_{i=1}^{k} a_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, b_i\} - \epsilon(k) \quad (\textit{CDZ-inequality})$$

holds for $0 \leq k \leq n$.

By choosing $a_i = b_i$ for $1 \leq i \leq n$, they showed that the Erdős–Gallai condition for a sequence of nonnegative integers to be graphic follows from their result. The main purpose of our paper is to give a constructive proof of the result in Theorem CDZ, along the lines of the one for the Erdős–Gallai Theorem that recently appeared in [7]. It is convenient to denote the inequality in Theorem CDZ as the CDZ-inequality. We begin by proving the following result, which is stronger than the necessary condition in Theorem CDZ.

Lemma 1. *Let* $\mathcal{A} := a_1, \ldots, a_n$ *and* $\mathcal{B} := b_1, \ldots, b_n$ *be two lists of nonnegative integers in good order such that* $a_i \leq b_i$ *for* $1 \leq i \leq n$. *Let G be a simple graph with vertices* $v_1, \ldots, v_n$ *such that* $a_i \leq d(v_i) \leq b_i$ *for* $1 \leq i \leq n$. *For* $J \subseteq \{1, \ldots, n\}$, *define*

$$I(J) := \{j \notin J : b_j \geq |J| + 1\}$$

and

$$\epsilon(J) = \begin{cases} 1 & \textit{if } a_j = b_j \textit{ for } j \in I(J) \textit{ and } \sum_{j \in I(J)} b_j + |J||I(J)| \textit{ is odd;} \\ 0 & \textit{otherwise.} \end{cases}$$

Then

$$\sum_{j \in J} a_j \leq |J|(|J| - 1) + \sum_{j \notin J} \min\{|J|, b_j\} - \epsilon(J). \tag{1}$$

Proof. If $J = \emptyset$ and $\epsilon(\emptyset) = 1$, then $I(\emptyset) = \{1, \ldots, n\}$, $a_i = d(v_i) = b_i$ for $1 \le i \le n$ and $\sum_{1\le i\le n} d(v_i)$ is odd. This contradiction proves that $\epsilon(\emptyset) = 0$, so that inequality (1) holds in this case. If $J \ne \emptyset$,

$$\sum_{j\in J} a_j \le \sum_{j\in J} d(v_j) \le |J|(|J|-1) + \sum_{j\notin J} \min\{|J|, d(v_j)\} \le |J|(|J|-1) + \sum_{j\notin J} \min\{|J|, b_j\},$$

the second inequality because the contribution from edges with both endpoints from J is no more than $|J|(|J|-1)$ and that from edges with exactly one endpoint from J is no more than either $|J|$ or $d(v_j)$. Thus inequality (1) holds unless $\epsilon(J) = 1$ for some J. If $\epsilon(J) = 1$, $a_i = d(v_i) = b_i$ for $i \in I(J)$ and $\sum_{j\in I(J)} d(v_j) + |J||I(J)|$ is odd. If inequality (1) does not hold for this J, then

$$\sum_{j\in J} d(v_j) = |J|(|J|-1) + \sum_{j\notin J} \min\{|J|, d(v_j)\}.$$

Then for v_i with $i \notin J \cup I(J)$, $N(v_i) \subseteq \{v_j : j \in J\}$. Moreover, each v_i with $i \in I(J)$ is adjacent to each v_j with $j \in J$ and not adjacent to any v_j with $j \notin J \cup I(J)$. Consider the subgraph G_0 induced by indices in $I(J)$. The sum of the degrees of vertices in G_0 is $\sum_{j\in I(J)} b_j - |J||I(J)|$. But this sum is odd because $\sum_{j\in I(J)} d(v_j) + |J||I(J)|$ is odd, and this is a contradiction. □

Remark 1. Choosing $J = \{1, \ldots, k\}$ in inequality (1) gives the CDZ-inequality. Thus the necessary condition in Theorem CDZ follows from Lemma 1.

Constructive proof of the sufficient condition in Theorem CDZ.

Consider two lists $\mathcal{A} := a_1, \ldots, a_n$ and $\mathcal{B} := b_1, \ldots, b_n$ of nonnegative integers in good order such that $a_i \le b_i$ for $1 \le i \le n$. Without loss of generality, we may further assume that $b_i \ge 1$ for $1 \le i \le n$. Call a graph G with vertices $v_1, \ldots, v_n$ a `sub-realization` if $d(v_i) \le b_i$ for all i, and a `realization` if $d(v_i) \ge a_i$ for all i. Define the `critical index` to be the largest index r such that $d(v_i) \ge a_i$ for $1 \le i < r$. We shall iteratively remove the deficiency $a_r - d(v_r)$ at vertex r without violating the range of degrees of previous vertices, by adding edges incident to v_r. Note that this involves two iterations: the inductive outer loop on r and the inductive inner loop on the deficiency $a_r - d(v_r)$. Initially we start with n vertices and no edges, so that $r = 1$, unless $a_i = 0$ for all i, in which case the process is complete. Let $S = \{v_{r+1}, \ldots, v_n\}$. We maintain the condition that S is an independent set. We consider the following cases:

- Case (0): $v_r \not\leftrightarrow v_i$ for some i with $d(v_i) < b_i$. Otherwise, for all i, *either* $v_r \not\leftrightarrow v_i$ and $d(v_i) = b_i$ *or* $v_r \leftrightarrow v_i$.
- Case (1): $v_r \not\leftrightarrow v_i$ for some $i < r$ and $d(v_i) = b_i$. Otherwise, $v_r \leftrightarrow v_i$ for $i \in \{1, \ldots, r-1\}$.
- Case (2): $v_r \leftrightarrow v_i$ for $i \in \{1, \ldots, r-1\}$ and $d(v_k) \ne \min\{r, b_k\}$ for some k with $k > r$. Otherwise, $v_r \leftrightarrow v_i$ for $i \in \{1, \ldots, r-1\}$ and $d(v_k) = \min\{r, b_k\}$ for all k with $k > r$.
- Case (3): $v_r \leftrightarrow v_i$ for $i \in \{1, \ldots, r-1\}$, $v_i \not\leftrightarrow v_j$ for some $i < j < r$ and $d(v_k) = \min\{r, b_k\}$ for all k with $k > r$. Otherwise, $\{v_1, \ldots, v_r\}$ is a clique and $d(v_k) = \min\{r, b_k\}$ for all k with $k > r$.
- Case (4): $\{v_1, \ldots, v_r\}$ is a clique, $d(v_k) = \min\{r, b_k\}$ for all k with $k > r$, and $d(v_i) > a_i$ for some $i \in \{1, \ldots, r-1\}$. Otherwise, $\{v_1, \ldots, v_r\}$ is a clique, $d(v_k) = \min\{r, b_k\}$ for all k with $k > r$ and $d(v_i) = a_i$ for $i \in \{1, \ldots, r-1\}$.

Case (0). If $v_r \not\leftrightarrow v_i$ for some i with $d(v_i) < b_i$, add the edge $v_r v_i$.

Case (1). Suppose $v_r \not\leftrightarrow v_i$ for some $i < r$ and $d(v_i) = b_i$. Note that there exists a vertex $u \in N(v_i) \setminus N(v_r)$, $u \ne v_r$, since $d(v_i) \ge a_i \ge a_r > d(v_r)$, and this holds for each $i \in \{1, \ldots, r-1\}$.

Choose $i < r$ such that $v_i \not\leftrightarrow v_r$. Suppose $b_r - d(v_r) \ge 2$. Replace $\{uv_i\}$ by $\{uv_r, v_i v_r\}$. This reduces the deficiency at r by 2. Since $a_r \le b_r$, the deficiency at r is also reduced by 2 if $a_r - d(v_r) \ge 2$. In particular, the deficiency at r reduces to 0 in one step if $a_r - d(v_r) = 2$.

Suppose $b_r - d(v_r) = 1$, so that $a_r = b_r$. If $v_r \leftrightarrow v_k$ for some $v_k \in S$, then delete $v_r v_k$ and apply the argument in the first part of this case. Otherwise, $v_r \not\leftrightarrow v_k$ for all $v_k \in S$. So Case (0) applies unless $d(v_k) = b_k$ for all $v_k \in S$. Let $S_1 := \{v_i : v_i \leftrightarrow v_r, 1 \le i \le r-1\}$ and $S_2 = \{v_1, \ldots, v_{r-1}\} \setminus S_1$. If $d(v_i) > a_i$ for some $i \in \{1, \ldots, r-1\}$, replace $\{uv_i\}$ by $\{uv_r\}$. If $d(v_i) < b_i$ for some $v_i \in S_2$, then add the edge $v_r v_i$. We may therefore assume that $d(v_i) = a_i$ for each $i \in \{1, \ldots, r-1\}$ and $d(v_i) = b_i$ for each $v_i \in S_2$.

If $v_i \leftrightarrow v_k$ for some $v_i \in S_2$ and $v_k \in S$, then replace $v_i v_k$ by $v_i v_r$. Hence we may assume that vertices in S_2 are adjacent only to vertices in $\{v_1, \ldots, v_{r-1}\}$.

Partition S_1 into X_1, X_2 and X_3 as follows:

$$X_1 = \{v_i : v_i \in S_1, (\exists v_k \in S)(v_i \leftrightarrow v_k)\}, \qquad X_2 = \{v_i : v_i \in S_1 \setminus X_1, a_i = b_i\}, \qquad X_3 = S_1 \setminus (X_1 \cup X_2).$$

We claim that if some vertex in $X_1 \cup X_3$ is not adjacent to $\{v_1, \ldots, v_r\}$, then we can decrease the deficiency at v_r.

If $v_i \not\leftrightarrow v_j$ for some $v_i \in X_1$ and $v_j \in S_1$, then $v_i \leftrightarrow v_k$ for some $v_k \in S$. Replace $\{v_i v_k, v_j v_r\}$ by $\{v_i v_j\}$. This increases the deficiency at v_r by 1, and the first part of Case (1) applies.

If $v_i \not\leftrightarrow v_j$ for some $v_i \in X_1$ and $v_j \in S_2$, then $v_i \leftrightarrow v_k$ for some $v_k \in S$. Note that $|S_1| = d(v_r) = a_r - 1 < a_j$. Hence $v_j \leftrightarrow v_\ell$ for some $v_\ell \in S_2$. Replace $\{v_i v_k, v_j v_\ell\}$ by $\{v_i v_j, v_\ell v_r\}$.

If $v_i \not\leftrightarrow v_j$ for some $v_i \in X_3$ and $v_j \in X_2 \cup X_3$, then replace $\{v_j v_r\}$ by $\{v_i v_j\}$. This increases the deficiency at v_r by 1, and the first part of Case (1) applies.

If $v_i \nleftrightarrow v_j$ for some $v_i \in X_3$ and $v_j \in S_2$, then $v_j \leftrightarrow v_\ell$ for some $v_\ell \in S_2$. Replace $\{v_j v_\ell\}$ by $\{v_i v_j, v_\ell v_r\}$.

In the last two cases, we increase $d(v_i)$ by 1; this can be done as $d(v_i) = a_i < b_i$, by definition of X_3. This completes the proof of the claim above.

We may therefore assume that $v_i \leftrightarrow v_j$ for $v_i \in X_1 \cup X_3$ and $1 \le j \le r$. We prove that this leads to a contradiction.

Let $K = \{i : v_i \in X_1 \cup X_3\}$, and let $k = |K|$. For $i \in K$ and $j \in \{1, \dots, r\}$, $v_i \leftrightarrow v_j$. For $v_\ell \in S$, the only vertices adjacent to v_ℓ are those in X_1. Hence $d(v_\ell) = b_\ell \le k$. For $j \in \{1, \dots, r\}$, $b_j \ge a_j \ge a_r \ge k + 1$, since $a_r - 1 \ge k$ as v_r is adjacent to all vertices in $X_1 \cup X_3$. Hence for $v_j \in X_2 \cup S_2$, $d(v_j) = b_j \ge k + 1$. It now follows that

$$\sum_{i \in K} a_i = k(k-1) + \sum_{i \notin K} \min\{k, b_i\}. \tag{2}$$

Define $f : \mathcal{P}(\{1, \dots, n\}) \to \mathbb{Z}$ by

$$f(J) = k(k-1) + \sum_{j \notin J} \min\{k, b_j\} - \sum_{j \in J} a_j - \epsilon(J).$$

We show that among k-subsets J of $\{1, \dots, r-1\}$, $f(J)$ is minimized when $J = \{1, \dots, k\}$. If J is a k-subset of $\{1, \dots, r-1\}$ and $J \ne \{1, \dots, k\}$, choose $j \in J \setminus \{1, \dots, k\}$ and $i \in \{1, \dots, k\} \setminus J$. If we denote by J' the set obtained from J on replacing j by i, then

$$f(J') - f(J) = \min\{k, b_j\} - \min\{k, b_i\} + \epsilon(J) - \epsilon(J') + a_j - a_i = \epsilon(J) - \epsilon(J') + a_j - a_i,$$

since $b_i, b_j \ge k + 1$. Thus $f(J') > f(J)$ is only possible when $\epsilon(J) = 1$, $\epsilon(J') = 0$, and $a_i = a_j$. Now $\epsilon(J) = 1$ implies $a_\ell = b_\ell$ for all $\ell \in I(J)$ and $\sum_{\ell \in I(J)} b_\ell + k|I(J)|$ is odd. Hence $b_i \ge b_j \ge a_j = a_i = b_i$, so that $a_j = b_j$ and $b_i = b_j$. Thus $\epsilon(J') = 1$, which is a contradiction. This proves the claim that $f(J)$ is minimized when $J = \{1, \dots, k\}$. Thus $f(K) \ge f(\{1, \dots, k\}) \ge 0$, by the CDZ-inequality.

Consider the graph G_1 induced by vertices in $S_2 \cup X_2 \cup \{v_r\}$. The sum of the degrees of vertices in G_1 equals $\sum_{v_j \in V(G_1)} d(v_j) - k|G_1| = \sum_{v_j \in V(G_1)} a_j - k|G_1| - 1$. Since this sum must be even, $\sum_{v_j \in V(G_1)} a_j - k|G_1|$ is odd. Also, $I(K)$ is equal to the set of indices in G_1 and $a_j = b_j$ for all $v_j \in V(G_1)$. Thus $\epsilon(K) = 1$, which contradicts (2) and $f(K) \ge 0$.

Case (2). Suppose $v_r \leftrightarrow v_i$ for $i \in \{1, \dots, r-1\}$ and $d(v_k) \ne \min\{r, b_k\}$ for some k with $k > r$. Since $d(v_k) \le b_k$ and S is an independent set, $d(v_k) \le r$ implies $d(v_k) < \min\{r, b_k\}$. Case (0) applies unless $v_r \leftrightarrow v_k$. Since $d(v_k) < r$, $v_k \nleftrightarrow v_i$ for some $i \in \{1, \dots, r-1\}$. Note that there is a vertex u such that $u \in N(v_i) \setminus (N(v_r) \cup \{v_r\})$. Replace $\{uv_i\}$ by $\{uv_r, v_k v_i\}$.

Case (3). Suppose $v_r \leftrightarrow v_i$ for $i \in \{1, \dots, r-1\}$, $v_i \nleftrightarrow v_j$ for some $i < j < r$ and $d(v_k) = \min\{r, b_k\}$ for all k with $k > r$. Then there exist vertices u and w, not necessarily distinct, such that $u \in N(v_i) \setminus (N(v_r) \cup \{v_r\})$ and $w \in N(v_j) \setminus (N(v_r) \cup \{v_r\})$. Since $v_1, \dots, v_{r-1} \in N(v_r)$, we have $u, w \in S$. Replace $\{uv_i, wv_j\}$ by $\{v_i v_j, uv_r\}$.

Case (4). Suppose $\{v_1, \dots, v_r\}$ is a clique, $d(v_k) = \min\{r, b_k\}$ for all k with $k > r$, and $d(v_i) > a_i$ for some $i \in \{1, \dots, r-1\}$. Then there exists a vertex $u \in N(v_i) \setminus (N(v_r) \cup \{v_r\})$. Replace $\{uv_i\}$ by $\{uv_r\}$.

If none of these cases apply, $v_1, \dots, v_r$ are pairwise adjacent and $d(v_k) = \min\{r, b_k\}$ for all $k > r$. Since S is independent set of vertices, we have $\sum_{i=1}^{r} d(v_i) = r(r-1) + \sum_{i=r+1}^{n} \min\{r, b_i\}$. Since $d(v_i) = a_i$ for $i < r$, we have

$$\sum_{i=1}^{r} a_i \le r(r-1) + \sum_{i=r+1}^{n} \min\{r, b_i\} = \sum_{i=1}^{r-1} a_i + d(v_r).$$

Hence $d(v_r) = a_r$. Increase r by 1 and continue. □

3. A constructive extension of the Gale–Ryser Theorem

The theorem of Gale and Ryser (Theorem GR) is the standard characterization of two lists of nonnegative integers that realize a bipartite graph. Following the result of Cai et al., it is natural to ask when two lists of intervals realize a bipartite graph in the same sense, as given in Section 1. We provide such a characterization, along lines similar to Theorem GR, and provide a constructive proof.

Theorem 1. *Let $\mathcal{I} := [a_1, b_1], \dots, [a_m, b_m]$ and $\mathcal{J} := [c_1, d_1], \dots, [c_n, d_n]$ be two lists of intervals consisting of nonnegative integers with $a_1 \ge \cdots \ge a_m$ and $c_1 \ge \cdots \ge c_n$. Then there exists a bipartite graph G with partite sets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$, and $a_i \le d(x_i) \le b_i$ for $1 \le i \le m$ and $c_i \le d(y_i) \le d_i$ for $1 \le i \le n$ if and only if*

$$\sum_{i=1}^{k} a_i \le \sum_{j=1}^{n} \min\{d_j, k\} \quad \text{for } 1 \le k \le m, \qquad \sum_{i=1}^{k} c_i \le \sum_{j=1}^{m} \min\{b_j, k\} \quad \text{for } 1 \le k \le n. \tag{3}$$

Proof. For the necessity part, suppose G is bipartite graph with partite sets X and Y satisfying the given conditions. Consider the edges incident to a set of k vertices in X. Each $y_j \in Y$ is incident to at most k of these vertices, and also incident to at most $d(y_j)$ of these vertices. So, for each $k \geq 1$,

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} d(x_i) \leq \sum_{j=1}^{n} \min\{d(y_j), k\} \leq \sum_{j=1}^{n} \min\{d_j, k\}.$$

The same argument applied to vertices in Y proves the other inequality.

For the sufficiency part, consider two lists of intervals $\mathcal{I} := [a_1, b_1], \ldots, [a_m, b_m]$ and $\mathcal{J} := [c_1, d_1], \ldots, [c_n, d_n]$ of nonnegative integers with $a_1 \geq \cdots \geq a_m$ and $c_1 \geq \cdots \geq c_n$, and satisfying the inequalities in (3). We first construct a bipartite graph G' with partite sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ satisfying $d(x_i) = a_i$ for $1 \leq i \leq m$ and $d(y_j) \leq d_j$ for $1 \leq j \leq n$. Define the `critical index` to be the largest index r such that $d(x_i) = a_i$ for $1 \leq i < r$ and $d(x_r) < a_r$. We shall iteratively remove the deficiency $a_r - d(x_r)$ at vertex x_r while maintaining $d(x_i) = a_i$ for $1 \leq i < r$ and $d(y_j) \leq d_j$ for $1 \leq j \leq n$. Initially we start with $m + n$ vertices and no edges, so that $r = 1$, unless $a_i = 0$ for all i, in which case the process is complete. Let $S = \{x_{r+1}, \ldots, x_m\}$. Note that there exists a vertex $v \in N(x_i) \setminus N(x_r)$ for $1 \leq i < r$, since $d(x_i) = a_i \geq a_r > d(x_r)$.

Case (1). Suppose, for some j, $y_j \leftrightarrow x_k$ for some $k > r$ and $y_j \not\leftrightarrow x_\ell$ for some $\ell \leq r$. If $\ell = r$, replace $\{x_k y_j\}$ by $\{x_r y_j\}$. If $\ell < r$, replace $\{x_k y_j, x_\ell v\}$ by $\{x_\ell y_j, x_r v\}$, where $v \in N(x_\ell) \setminus N(x_r)$.

Case (2). Suppose, for some j, $d(y_j) < d_j$ and $y_j \not\leftrightarrow x_\ell$ for some $\ell \leq r$. If $\ell = r$, add the edge $x_r y_j$. If $\ell < r$, replace $\{x_\ell v\}$ by $\{x_\ell y_j, x_r v\}$, where $v \in N(x_\ell) \setminus N(x_r)$.

If none of the cases above arise, then

$$\sum_{i=1}^{r-1} a_i + d(x_r) = \sum_{i=1}^{r} d(x_i) = \sum_{j=1}^{n} \min\{d(y_j), r\} = \sum_{j=1}^{n} \min\{d_j, r\}, \tag{4}$$

since $d(v_i) = a_i$ for $1 \leq i \leq r - 1$, and from (3) we have $d(x_r) = a_r$. Increasing r by 1, and applying the same steps leads to the required bipartite graph G'.

We complete the construction by applying the same argument to the bipartite graph G', with partite sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ satisfying $d(x_i) = a_i$ for $1 \leq i \leq m$ and $d(y_j) \leq d_j$ for $1 \leq j \leq n$. We define the `critical index` to be the largest index s such that $d(y_j) \geq c_j$ for $1 \leq j < s$ and $d(y_s) < c_s$. We decrease the deficiency at vertex y_r while maintaining $d(y_j) \geq c_j$ for $1 \leq j < r$ and $a_i \leq d(x_i) \leq b_i$ for $1 \leq i \leq m$, except that an additional case arises.

Case (3). If $d(y_i) > c_i$ for some $i < r$, then replace $\{y_i v\}$ by $\{y_r v\}$, where $v \in N(y_i) \setminus N(y_r)$.

If none of the three cases arise, then similar to Eq. (4), we again arrive at $d(y_r) = c_r$. The argument above now leads to a construction for G'. □

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