An Alternate Method to Compute the Decimal Expansion of Rational Numbers

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Every middle school student knows how to use long division to find the decimal expansion of a rational number. Doing so results in writing the digits of the decimal expansion from left to right. Is there a way to write the decimal expansion from right to left? The purpose of this note is to describe a procedure to write such a decimal expansion. While a high school student in 1976, I was aware that this procedure applied to rational numbers whose denominators were primes ending in 9. It wasn't until 2006 that I was surprised to discover that the procedure could be extended to all denominators.

Since we are interested in the decimal expansion of rational numbers, we may suppose that a/n is a rational number, in reduced form. To implement our procedure, we follow the two steps listed below.

- **Step 1.** If $\gcd(n, 10) > 1$, write a/n in terms of b/m with $\gcd(m, 10) = 1$. If $\gcd(n, 10) = 1$, then define $a = b$ and $m = n$.

As an example of Step 1, consider the decimal expansion of $r = 7/760$ so that $r = 7/(2^3 \cdot 5 \cdot 19)$. From

$$r = \frac{1}{10^3}\,\frac{5^2 \cdot 7}{19} = \frac{1}{10^3}\left(9 + \frac{4}{19}\right), \tag{1}$$

we see that the decimal expansion of r can be determined from that of $4/19$. In this example, $n = 2^3 \cdot 5 \cdot 19$ and $m = 19$, and we have linked the decimal expansion of a rational number a/n with $\gcd(n, 10) > 1$ to the decimal expansion of b/m with $\gcd(m, 10) = 1$. So when $\gcd(n, 10) > 1$, Step 1 requires us to express a/n as $b/(m \cdot 10^\gamma)$ for an appropriate choice of γ. If $\gcd(a, n) = 1$, we let $a = b$ and $m = n$, and move to Step 2.

- **Step 2.** The decimal expansion of b/m, $\gcd(m, 10) = 1$ is always a purely recurring decimal. To write down the digits from right to left, determine the last digit, a positive integer c by which to multiply each digit to get the subsequent digit, and the number of digits.

Let us illustrate Step 2 by continuing with the example in Step 1 – the decimal expansion of $4/19$. It is well known that the decimal expansion will be purely recurring. To write down the digits from right to left in a sequence, we determine (i) the starting digit (the last digit in the expansion), (ii) the positive integer c (we call this the multiplying factor), and (iii) the number of digits in the recurring part. Theorem 1 shows that we must start the sequence with 4, use the multiplying factor 2, and stop after 18 digits. Therefore, we begin with 4, then multiply 4 by 2 (the multiplying factor

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for this example) to get 8 as the second digit (from right to left!). To proceed further, we multiply 8 by 2 to get 16; this gives 6 as the third digit and a carry-over 1.

To get the fourth digit from the third, we multiply 6 by 2 and add any carry-over that may have accrued. This gives 13; so 3 is the fourth digit and we again carry-over 1. A careful and patient application of this procedure results in the following sequence; the carry-overs appear in bold face above the corresponding the digits.

$$4, 8, 6^{(\mathbf{1})}, 3^{(\mathbf{1})}, 7, 4^{(\mathbf{1})}, 9, 8^{(\mathbf{1})}, 7^{(\mathbf{1})}, 5^{(\mathbf{1})}, 1^{(\mathbf{1})}, 3, 6, 2^{(\mathbf{1})}, 5, 0^{(\mathbf{1})}, 1, 2.$$

Thus

$$\frac{4}{19} = 0.\overline{210526315789473684}. \tag{2}$$

Finally, substituting (2) in (1):

$$\frac{7}{2^3 \cdot 5 \cdot 19} = \frac{1}{10^3}\left(9 + \frac{4}{19}\right) = 0.009\,\overline{210526315789473684}.$$

Suppose $a/n \in \mathbb{Q}$, with $1 \leq a < n$, $\gcd(a, n) = 1$, $n = 2^\alpha 5^\beta m$, and $\gcd(m, 10) = 1$. Then the decimal expansion of a/n terminates if $m = 1$ and recurs if $m > 1$. In the latter case, if ℓ denotes the number of digits in the recurring part and γ the number of digits in the nonrecurring part, ℓ equals the order of 10 modulo m and $\gamma = \max\{\alpha, \beta\}$. The statement ℓ equals the order of 10 modulo m means that ℓ is the least positive integer for which m divides $10^\ell - 1 = \underbrace{9 \cdots 9}_{\ell \text{ times}}$; we denote this by $\ell = \operatorname{ord}_m 10$. Given a positive integer m, we know that $\operatorname{ord}_m b$ exists if and only if $\gcd(b, m) = 1$, or in other words, when b is an element of the multiplicative group of units in the ring $\mathbb{Z}_m$. A proof of the statement that $\ell = \operatorname{ord}_m 10$ can be found in [**1**, pp. 109–111], but we give a short and self-contained proof nevertheless.

The decimal expansion of a/n terminates if and only $10^e a/n$ is an integer, which is true if and only if n divides 10^e, so that $m = 1$. Assume now that $n = 2^\alpha 5^\beta m$, with $\gcd(m, 10) = 1$ and $m > 1$. If we reduce each term of the infinite sequence $\{a10^i\}_{i \geq 1}$ mod n, by the *pigeonhole principle*, two terms must be equal. If we choose the first such pair, say $a10^s$ and $a10^{s+\ell}$, then $10^s \equiv 10^{s+\ell} \mod n$, since $\gcd(a, n) = 1$. From $\gcd(m, 10) = 1$ it follows that $10^\ell \equiv 1 \mod m$. Since this implies $a10^{s+k} \equiv a10^{s+\ell+k}$ mod n for each $k \geq 0$, the sequence is eventually recurring and the length of the period equals $\ell = \operatorname{ord}_m 10$. From $10^s \equiv 10^{s+\ell} \mod n$ we see that both 2^α and 5^β must divide 10^s since each is coprime to $10^\ell - 1$. The smallest such s equals $\gamma = \max\{\alpha, \beta\}$, and this represents the length of the nonrecurring part.

We may suppose a/n is a positive rational number, in reduced form and less than 1, for the purpose of decimal expansion. Let $n = 2^\alpha 5^\beta m$, where $\gcd(m, 10) = 1$, and set $\gamma = \max\{\alpha, \beta\}$. Since

$$10^\gamma \frac{a}{n} = 2^{\gamma-\alpha} 5^{\gamma-\beta} \frac{a}{m} = \frac{b}{m},$$

the decimal expansion of a/n is the same as that of b/m but with the decimal shifted γ places to the left. The general case is therefore easily reduced to the case where $\gcd(n, 10) = 1$. For example, to obtain the decimal expansion of $7/(2^3 \cdot 5 \cdot 19)$, we have already noted that

$$\frac{7}{2^3 \cdot 5 \cdot 19} = \frac{1}{10^3} \frac{5^2 \cdot 7}{19} = \frac{1}{10^3}\left(9 + \frac{4}{19}\right).$$

We now consider rational numbers with denominator coprime to 10. The decimal expansion of such rational numbers is purely recurring, and the digits when sequenced from right to left satisfy a recurrence equation.

Theorem 1. *Let a, n be positive integers, with $1 \le a < n$ and $\gcd(a, n) = \gcd(n, 10) = 1$. Then*

$$\frac{a}{n} = 0.\overline{a_1\, a_2\, a_3 \cdots a_\ell},$$

where $\ell = \text{ord}_n\, 10$, and the finite sequence $\{a_k\}_{k=1}^{\ell}$ satisfies the recurrence

$$a_k \equiv \frac{10c-1}{10}a_{k+1} + \frac{10c-1}{10^2}a_{k+2} + \cdots + \frac{10c-1}{10^{\ell-k-1}}a_{\ell-1} + \frac{an^\star}{10^{\ell-k}} \pmod{10}$$

for $1 \le k \le \ell - 1$, with the initial condition

$$a_\ell \equiv an^\star \pmod{10},$$

where $c \equiv 10^{-1} \mod n$, $1 \le c \le n-1$ and $n^\star \equiv -n^{-1} \mod 10$, $1 \le n^\star \le 9$.

Proof. Write $a/n = 0.\overline{a_1\, a_2\, a_3 \cdots a_\ell}$; we know that $\ell = \text{ord}_n\, 10$. Let A_k denote the k-digit number $a_1 a_2 \cdots a_k$ for $k \ge 1$. Since $A_k = \lfloor a10^k/n \rfloor$, we have

$$a10^k = nA_k + r_k, \tag{3}$$

with $1 \le r_k < n$. Since a/n is the sum of an infinite geometric series with first term $A_\ell/10^\ell$ and common ratio $1/10^\ell$, we have

$$nA_\ell = a(10^\ell - 1). \tag{4}$$

Thus $na_\ell \equiv nA_\ell \equiv -a \pmod{10}$, and $a_\ell \equiv -an^{-1} \equiv an^\star \pmod{10}$.

Define c such that

$$10c \equiv 1 \pmod{n},$$

with $1 \le c \le n-1$. Since $n \cdot \frac{10c-1}{n} \equiv -1 \pmod{10}$ and both $\frac{10c-1}{n}$ and $n^\star$ lie between 1 and 9, it follows that

$$10c - 1 = nn^\star. \tag{5}$$

From (3), for $k > 1$,

$$r_{k-1} \equiv a10^{k-1} \equiv ac10^k \equiv cr_k \pmod{n}.$$

In particular, since $r_\ell = a$ by reducing (3) mod n, we have $r_{\ell-1} \equiv ac \pmod{n}$.

From (3), we have

$$10r_{k-1} - r_k = 10(a10^{k-1} - nA_{k-1}) - (a10^k - nA_k) = n(A_k - 10A_{k-1}) = na_k \tag{6}$$

valid for $2 \le k \le \ell$.

For each k, $1 \le k \le \ell - 1$, let us denote the $(\ell - k)$-digit number with digits $a_{k+1}a_{k+2} \cdots a_\ell$ by A'_k. Thus

$$A_\ell = 10^{\ell-k}A_k + A'_k,$$

so that by (3) and (4),

$$\begin{aligned} nA'_k + a &= n(A_\ell - 10^{\ell-k}A_k) + a = a10^\ell - n10^{\ell-k}A_k = 10^{\ell-k}(a10^k - nA_k) \\ &= 10^{\ell-k}r_k. \end{aligned} \tag{7}$$

From (5), (6), and (7),

$$\frac{(10c-1)A'_k + an^\star}{10^{\ell-k}} = \frac{n^\star(nA'_k + a)}{10^{\ell-k}} = n^\star r_k \equiv a_k \mod 10. \tag{8}$$

To complete the proof, replace A'_k by $a_{k+1}10^{\ell-k-1} + a_{k+2}10^{\ell-k-2} + \cdots + a_\ell$ in (8). ■

Remark. Let n be a positive integer such that $\gcd(n, 10) = 1$. Then

$$n^\star \equiv \begin{cases} -n \bmod 10 & \text{if } n \equiv \pm 1 \pmod{10}, \\ n \bmod 10 & \text{if } n \equiv \pm 3 \pmod{10}, \end{cases}$$

and

$$c \equiv \begin{cases} \dfrac{9n+1}{10} & \text{if } n \equiv 1 \pmod{10}, \\ \dfrac{3n+1}{10} & \text{if } n \equiv 3 \pmod{10}, \\ \dfrac{7n+1}{10} & \text{if } n \equiv 7 \pmod{10}, \\ \dfrac{n+1}{10} & \text{if } n \equiv 9 \pmod{10}. \end{cases}$$

Proof. We observe that $\gcd(n, 10) = 1$ is equivalent to stating that $n \equiv \pm 1$ or ± 3 modulo 10. Since $n^2 \equiv 1 \pmod{10}$ when $n \equiv \pm 1 \pmod{10}$ and $n^2 \equiv -1 \pmod{10}$ when $n \equiv \pm 3 \pmod{10}$, we have $n^{-1} \equiv n \mod 10$ in the first case and $n^{-1} \equiv -n \pmod{10}$ in the second case. This leads to the results on $n^\star$.

Recall that c is the unique integer satisfying $10c \equiv 1 \pmod{n}$ and $1 \le c \le n-1$. It is easy to verify that the case-by-case values of c given above satisfy both conditions and are integers. This completes the proof. ■

Let us revisit Step 2 of the procedure to determine the decimal expansion of $4/19$ in light of Theorem 1. Theorem 1 applies only to cases where n ends in 1, 3, 7, or 9, and so Step 2 applies to only those cases. For our example, $n^\star = 1$ and $c = 2$ by Remark. Thus $a_\ell \equiv an^\star \equiv a \pmod{10}$, and so $a_\ell = 4$ is the first term in the sequence. The number of terms in the sequence is given by $\ell = \operatorname{ord}_n 10$, as remarked earlier. Recall that ℓ is the least positive integer for which $10^\ell - 1 = \underbrace{9\cdots9}_{\ell \text{ times}}$ is a multiple of n. There is no simple way to compute ℓ, but because 10 is an element of the multiplicative group of units in the ring $\mathbb{Z}_n$ we know that $\ell \mid \phi(n)$.

The following reformulation of the recurrence in Theorem 1 leads to a computationally interesting consequence:

$$a_k \equiv ca_{k+1} + \left(\frac{ca_{k+2} - a_{k+1}}{10} + \cdots + \frac{ca_\ell - a_{\ell-1}}{10^{\ell-k-1}} + \frac{an^\star - a_\ell}{10^{\ell-k}}\right) \pmod{10} \tag{9}$$

for $1 \le k \le \ell - 2$, with

$$a_\ell \equiv an^\star \mod 10, \quad a_{\ell-1} \equiv ca_\ell + \frac{an^\star - a_\ell}{10} \pmod{10}.$$

For each k, $1 \le k \le \ell - 2$, let us denote by λ_{k+1} the expression within brackets in (9), and let $\lambda_\ell = (an^\star - a_\ell)/10$. For $1 \le k \le \ell - 1$, set

$$ca_{k+1} + \lambda_{k+1} = 10\mu_k + a_k.$$

Then

$$\mu_k = \frac{ca_{k+1} - a_k}{10} + \frac{ca_{k+2} - a_{k+1}}{10^2} + \cdots + \frac{ca_\ell - a_{\ell-1}}{10^{\ell-k}} + \frac{an^\star - a_\ell}{10^{\ell-k+1}} = \lambda_k,$$

so that

$$ca_{k+1} + \lambda_{k+1} = 10\lambda_k + a_k$$

for $1 \le k \le \ell - 2$.

We summarize the procedure for the computation of the sequence of digits $a_\ell, a_{\ell-1}, \ldots, a_1$ that appear in the decimal expansion of a/n when $\gcd(n, 10) = 1$ in our concluding remark.

Remark. Let a, n be positive integers, with $1 \le a < n$ and $\gcd(a, n) = \gcd(n, 10) = 1$. Then $a/n = 0.\overline{a_1\, a_2\, \cdots\, a_\ell}$, where $\ell = \mathrm{ord}_n\, 10$. Define two positive integers $n^\star$ and c as follows:

$$10 \mid (nn^\star + 1), \qquad c = \frac{nn^\star + 1}{10},$$

where $1 \le n^\star \le 9$.
The sequence $\{a_\ell, a_{\ell-1}, \ldots, a_1\}$ and the sequence $\{\lambda_\ell, \lambda_{\ell-1}, \ldots, \lambda_1\}$ are interlinked. The first sequence consists of integers in [0, 9] and satisfies (9):

$$a_k \equiv ca_{k+1} + \lambda_{k+1} \mod 10 \tag{10}$$

for $1 \le k \le \ell - 1$, with

$$a_\ell \equiv an^\star \mod 10, \qquad \lambda_\ell = \frac{an^\star - a_\ell}{10}. \tag{11}$$

The second sequence consists of nonnegative integers and is given by the expression within brackets in (9):

$$\lambda_{k+1} = \frac{ca_{k+2} - a_{k+1}}{10} + \cdots + \frac{ca_\ell - a_{\ell-1}}{10^{\ell-k-1}} + \frac{an^\star - a_\ell}{10^{\ell-k}} \tag{12}$$

for $1 \le k \le \ell - 1$, with $\lambda_\ell = (an^\star - a_\ell)/10$. The two sequences are connected via the formulae:

$$\begin{aligned} an^\star &= 10\lambda_\ell + a_\ell, \\ ca_\ell + \lambda_\ell &= 10\lambda_{\ell-1} + a_{\ell-1}, \\ ca_{\ell-1} + \lambda_{\ell-1} &= 10\lambda_{\ell-2} + a_{\ell-2}, \\ &\vdots\ \vdots\ \vdots \\ ca_{k+1} + \lambda_{k+1} &= 10\lambda_k + a_k, \\ &\vdots\ \vdots\ \vdots \\ ca_2 + \lambda_2 &= 10\lambda_1 + a_1. \end{aligned}$$

When $\gcd(n, 10) = 1$, the digits in the decimal expansion of a/n are obtained recursively by computing the sequence of digits is $a_\ell, a_{\ell-1}, \ldots, a_1$. The rightmost digit a_ℓ is $an^\star \bmod 10$, the multiplying factor c equals $(nn^\star + 1)/10$, and the rest of the sequence $a_{\ell-1}, \ldots, a_1$ is computed via the sequence $\lambda_\ell, \lambda_{\ell-1}, \ldots, \lambda_1$ of carry-overs.

We close this article by applying the result in Remark to a special example. Repunits are numbers like $1, 11, 111, \ldots$ that contain only the digit 1; the ℓ-digit repunit equals $(10^\ell - 1)/9$. Repdigits are numbers like $d, dd, dd, \ldots$ that contain only the digit d; the ℓ-digit repdigit containing only the digit d equals $(10^\ell - 1)d/9$. The decimal expansion of a/n, $\gcd(a, n) = 1$, when $n = 10^\ell - 1$ has a particularly simple form. We may assume, as before, that $1 \le a < 10^\ell - 1$. With $a = d_1 10^{\ell-1} + d_2 10^{\ell-2} + \cdots + d_\ell$, and assuming $\gcd(a, n) = 1$, we use Remark to show that $a/n = \overline{d_1\, d_2\, d_3\, \cdots\, d_\ell}$. Note that if a has k digits and $k < \ell$, then $d_i = 0$ for $1 \le i \le \ell - k$.

We know that the decimal expansion of a reduced rational number a/n is purely recurring, and that the length of the recurring part equals $\text{ord}_n\, 10$. It is easy to see directly from the definition of order that $\text{ord}_n\, 10 = \ell$ when $n = 10^\ell - 1$. Thus a/n is of the form $\overline{a_1\, a_2\, a_3\, \cdots\, a_\ell}$, and we must show that $a_i = d_i$ for $1 \le i \le \ell$.

From Remark we get $n^\star = 1$ and $c = (n + 1)/10 = 10^{\ell-1}$. Hence $a_\ell = d_\ell$ and $\lambda_\ell = A_{\ell-1}$ by (11). Suppose $a_i = d_i$ for $i \in \{k + 1, \ldots, \ell\}$. Then from (10), (12) and the induction hypothesis, we get

$$\begin{aligned} a_k &\equiv -\frac{d_{k+1}}{10} - \cdots - \frac{d_{\ell-1}}{10^{\ell-k-1}} + \frac{a - d_\ell}{10^{\ell-k}} \pmod{10} \\ &= \frac{a - (d_{k+1} 10^{\ell-k-1} + \cdots + d_{\ell-1} 10 + d_\ell)}{10^{\ell-k}} \pmod{10} \\ &\equiv d_k \pmod{10}. \end{aligned}$$

This proves our claim by induction.

All this can be seen more directly by observing that the sum of the infinite geometric series with both first term and common ratio equal to $1/10^\ell$ is $1/(10^\ell - 1)$. So

$$\frac{a}{10^\ell - 1} = \frac{a}{10^\ell} + \frac{a}{10^{2\ell}} + \frac{a}{10^{3\ell}} + \cdots = \overline{a_1\, a_2\, a_3\, \cdots\, a_\ell},$$

where $a_1 a_2 a_3 \cdots a_\ell$ represents the number a. As before, if a has k digits and $k < \ell$, then $a_i = 0$ for $1 \le i \le \ell - k$. This immediately leads to the decimal expansion of a/n when $1 \le a < n$ and $\gcd(a, n) = \gcd(n, 10) = 1$. We know that $n \mid (10^\ell - 1)$ for some positive integer ℓ; write $nd = 10^\ell - 1$. This gives $a/n = ad/(10^\ell - 1)$ with $1 \le ad < 10^\ell - 1$, and we can immediately write down the decimal expansion of a/n.

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REFERENCE

1. G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers.* Fourth ed. Clarendon Press, Oxford Univ. Press, New York, 1959.

Summary. Rational numbers, except those with denominators of the form $2^a 5^b$, have a recurring decimal expansion. It is usual to write these digits from left to right. We give a procedure to write the decimal digits from the opposite end—right to left.

AMITABHA TRIPATHI (MR Author ID: 262251) enjoys lecturing on mathematical topics that are not routine, in particular mathematical problem solving. Outside of mathematics, he enjoys playing table tennis, working on crosswords and sudoku, listening to music, and when given an opportunity, demonstrating his computational abilities.