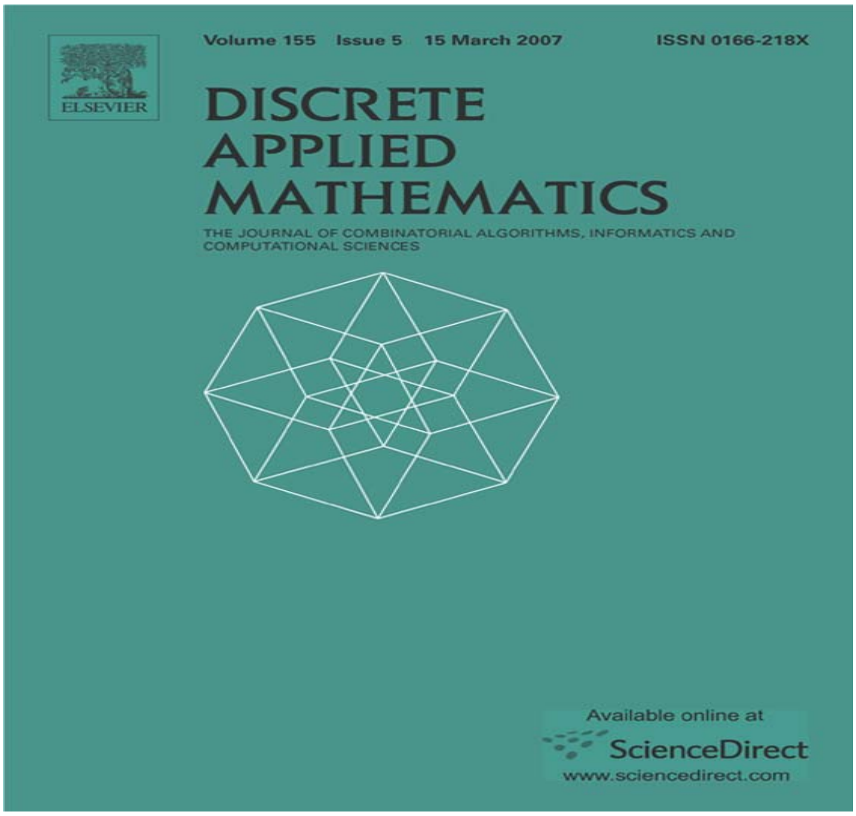
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Note

A short proof of a theorem on degree sets of graphs

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Abstract

For a finite simple graph G, we denote the set of degrees of its vertices, known as its *degree set*, by $\mathscr{D}(G)$. Kapoor, Polimeni and Wall [Degree sets for graphs, Fund. Math. 95 (1977) 189–194] have determined the least number of vertices among graphs with a given degree set. We give a very short proof of this result.

Keywords: Degree sequences; Degree sets

Let G be a finite simple graph, with n vertices. The degree of a vertex v, which we write as deg v, is the number of vertices of G adjacent to v. Since there can be at most one edge between any pair of vertices in a simple graph, deg $v \leqslant n-1$ for each vertex v. One of the most basic results in Graph Theory, which is also easy to prove, is that if we sum the degrees of vertices of a finite simple graph, the sum equals twice the number of edges in the graph; see [1], for instance.

The *degree sequence* of a finite simple graph G is the sequence of degrees of vertices of G. A sequence $a_1, a_2, \ldots, a_n$ of nonnegative integers is called *graphic* if it is the degree sequence of some finite simple graph. Any graphic sequence must satisfy the two conditions:

(a) $a_i \leqslant n-1$ for each i;
(b) $\sum_{i=1}^{n} a_i$ is *even*.

However, these two conditions together do not ensure that a sequence will be graphic; for instance, the sequence $\{3, 3, 3, 1\}$ is not graphic. Necessary and sufficient conditions for a sequence of nonnegative integers to be graphic are well-known. Two characterizations of graphic sequences are due to Havel [4], and later and independently by Hakimi [3], and by Erdös and Gallai [2].

For a simple graph G, we denote the set of degrees of its vertices, known as its *degree set*, by $\mathscr{D}(G)$. Kapoor et al. [5] have determined the least number of vertices among graphs with a given degree set.

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Theorem KPW (*Kapoor et al.* [5]). *For each finite set of positive integers S with largest element Δ, there exists a graph G for which $\mathscr{D}(G) = S$. Moreover, the least order of such a graph G with degree set S is $\Delta + 1$.*

In this note we give a short proof of this theorem. Henceforth, for a given finite set of positive integers S with greatest element Δ, let G_S be any graph whose degree set is S. Let us denote the number of vertices in G_S by $n(G_S)$, and by $\ell(S)$ the *minimum* number of vertices among G_S graphs. We give a short proof of Theorem KPW, that $\ell(S) = \Delta + 1$.

Proof of Theorem KPW. If S is the degree set of a graph G with p vertices, then the existence of a vertex with degree Δ necessitates $p \geqslant \Delta + 1$. To complete the proof, we prove the existence of a graph with $\Delta + 1$ vertices whose degree set is S.

We induct on the number of elements in S. If $S = \{a_1\}$, then $G_S = \mathscr{K}_{a_1+1}$, the complete graph with $a_1 + 1$ vertices, is the only possibility. Assume the result for all sets with fewer than n elements, and consider $S = \{a_1, a_2, \ldots, a_{n+1}\}$, arranged in decreasing order. For $T = \{a_1 - a_{n+1}, a_1 - a_n, \ldots, a_1 - a_2\}$, by induction hypothesis, there exists a graph G_T with $a_1 - a_{n+1} + 1$ vertices and whose degree set is T. Add a_{n+1} isolated vertices, those that have degree 0, to this graph. The complement of the resulting graph has $\Delta + 1$ vertices and has degree set S. □

It is an interesting fact that it is possible to construct a graphic sequence of minimal order with a given degree set [6], thereby providing a direct proof of Theorem KPW.

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