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On the density of integral sets with missing differences from sets related to arithmetic progressions

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ABSTRACT

For a given set M of positive integers, a problem of Motzkin asks for determining the maximal density $\mu(M)$ among sets of nonnegative integers in which no two elements differ by an element of M. The problem is completely settled when $|M| \leqslant 2$, and some partial results are known for several families of M for $|M| \geqslant 3$, including the case where the elements of M are in arithmetic progression. We consider some cases when M either contains an arithmetic progression or is contained in an arithmetic progression.

1. Introduction

For $x \in \mathbb{R}$ and a set S of nonnegative integers, let $S(x)$ denote the number of elements $n \in S$ such that $n \leqslant x$. The upper and lower densities of S, denoted by $\bar{\delta}(S)$ and $\underline{\delta}(S)$ respectively, are given by

$$\bar{\delta}(S) := \limsup_{x\to\infty} \frac{S(x)}{x}, \qquad \underline{\delta}(S) := \liminf_{x\to\infty} \frac{S(x)}{x}.$$

If $\bar{\delta}(S) = \underline{\delta}(S)$, we denote the common value by $\delta(S)$, and say that S has density $\delta(S)$. Given a set of positive integers M, S is said to be an M-set if $a \in S$, $b \in S$ imply $a - b \notin M$. Motzkin in [16] asked to determine $\mu(M)$ given by

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$$\mu(M) := \sup_S \bar{\delta}(S)$$

where S varies over all M-sets. Cantor and Gordon in [3] showed the existence of $\mu(M)$ for any M, determined $\mu(M)$ when $|M| \leqslant 2$, and gave the following lower bound for $\mu(M)$:

$$\mu(M) \geqslant \kappa(M) := \sup_{\gcd(x,m)=1} \frac{1}{m} \min_i |xm_i|_m, \tag{1}$$

where m_i are the elements of M, and $|x|_m$ denotes the absolute value of the absolutely least remainder of x modulo m. A useful upper bound for $\mu(M)$ is due to Haralambis in [11]:

$$\mu(M) \leqslant \alpha \tag{2}$$

provided there exists a positive integer k such that $S(k) \leqslant (k+1)\alpha$ for every M-set S with $0 \in S$ and for some $\alpha \in [0,1]$.

The problem of Motzkin has a rich and diverse history but little progress towards the general problem has been made so far. Exact results for $\mu(M)$ have been few, and computation of $\mu(M)$ has only been completely possible when $|M| \leqslant 2$; Cantor and Gordon in [3] showed that

$$\mu(\{m\}) = \frac{1}{2}, \qquad \mu(\{m_1, m_2\}) = \frac{\lfloor (m_1+m_2)/2 \rfloor}{m_1+m_2}.$$

There have, however, been a number of results that give the exact value or bounds for $\mu(M)$ in other cases; see [14,17] for an exhaustive bibliography.

Connections with colouring problems in Graph Theory have been found useful in solving the Motzkin problem. One such connection, introduced by Hale in [10] and shown to be equivalent to the problem by Griggs and Liu in [8], is the *T-colouring problem*. For a given set of nonnegative integers T with $0 \in T$, a T-colouring of a finite simple graph G with vertex set V is a function $f: V \to \mathbb{N} \cup \{0\}$ such that $|f(u) - f(v)| \notin T$ whenever $uv \in E(G)$. Rabinowitz and Proulx in [18] showed that if $\sigma_n := \min_f \{\max_{u,v \in V} |f(u) - f(v)|\}$ denotes the optimal span of the T-colourings of complete graphs K_n, then $rt(T) := \lim_{n\to\infty} n/\sigma_n$ exists, and Griggs and Liu in [8] then showed that $rt(T) = \mu(T \setminus \{0\})$; also see [7,12].

The other connection with colourings of graphs involves the *fractional chromatic number* of *distance graphs*. For a given set of positive integers M, the distance graph $G(\mathbb{Z}, M)$ generated by M, is the graph with vertex set $\mathbb{Z}$ and vertices x, y adjacent precisely when $|x - y| \in M$. A fractional colouring of G is a mapping f which assigns to each independent set I of vertices a nonnegative weight $f(I)$ such that $\sum_{x \in I} f(I) \geqslant 1$ for each vertex x, and the fractional chromatic number $\chi_f(G)$ of G is the least total weight of a fractional colouring f of G. Chang, Liu and Zhu in [5] showed that $\mu(M) \cdot \chi_f(G(\mathbb{Z}, M)) = 1$, thereby establishing the connection between fractional chromatic number of distance graphs and the Motzkin problem; also see [4,14,15].

The lower bound for $\mu(M)$, denoted by $\kappa(M)$ in (1), is itself at the heart of a longstanding conjecture. The *Lonely Runner Conjecture* (LRC) stated independently by Wills in [19] in the context of diophantine approximations and by Cusick in [6] while studying view obstructions problems in n-dimensional geometry, was actually given this apt name by Bienia et al. in [1]. The problem can be stated as follows.

> Consider $k+1$ runners on a circular track of unit length, all of whom start at the same point and time, and run at pairwise distinct constant speeds in the same direction. A runner is lonely at some point of time if she is at a distance at least $1/(k+1)$ along the track from every other runner. LRC states that every runner gets lonely at some point in time.

A convenient and usual reformulation of the LRC can be obtained by assuming that all speeds are integers, not all divisible by the same prime, and that the runner to be lonely has zero speed. If $\|x\|$ denotes the distance of the real number x to its nearest integer, then LRC states that, for any set D of k positive integers, there is a real number t such that $\|td\| \geqslant 1/(k+1)$ for each $d \in D$. Barajas and Serra in [2] established LRC for seven runners, and we refer to their work for more on the LRC.

One of the few general cases where $\mu(M)$ has been exactly determined is the case where the elements of M are in arithmetic progression. If we write $\mathscr{A} = \{a, a+d, \ldots, a+(n-1)d\}$, where $\gcd(a,d)=1$, Gupta and Tripathi in [9] showed that

$$\mu(\mathscr{A}) = \kappa(\mathscr{A}) = \begin{cases} \frac{1}{2} & \text{if } d \text{ is even;} \\ \frac{2a+(n-1)(d-1)}{2(2a+(n-1)d)} & \text{if } d \text{ is odd.} \end{cases}$$

Exact results for $\mu(M)$ when M is of the form $[1,n] \setminus [a,b]$ was also found as a result of the combined efforts of Wu and Lin in [20], Lam and Lin in [13], and Liu and Zhu in [15]. We state this for ready reference, albeit with notations consistent with what we use in Section 3.

Let n_1, n_2, k be positive integers, and let $n = n_1 + n_2 + k$. Let $s = \lfloor \frac{n_1+k}{n_1+1} \rfloor$ and $q = \lfloor \frac{n_1+k}{n_2+1} \rfloor + 1$. Let $M = [1, n_1] \cup [n_1+k+1, n]$.

(i) If $n_2 \leqslant n_1$, then $\mu(M) = \max\{n_1+1, \frac{n+1}{s+1}\}$.
(ii) If $n_1 < n_2 < 2n_1+1$, then

$$\mu(M) = \begin{cases} \frac{n+1}{q} & \text{if } n < \min\{q(n_2+1)+n_1, 2q(n_1+1)-1\}; \\ \frac{(2q-1)(n+1)+(n_1+1)}{2q^2} & \text{if } q(n_2+1)+n_1 \leqslant n < 2q(n_1+1)+n_1; \\ \frac{s(n_1+1)+(n+1)}{s+1} & \text{if } 2q(n_1+1)-1 \leqslant n < q(n_2+1)+n_1 \text{ or } n \geqslant 2q(n_1+1)+n_1. \end{cases}$$

(iii) If $n_2 \geqslant 2n_1+1$, then $\mu(M) = \frac{s(n_1+1)+(n+1)}{s+1}$.

In this paper, we explore the two related problems of determining $\mu(M)$: (i) when M contains $\mathscr{A}$; (ii) when M is contained in $\mathscr{A}$. The second problem arises naturally as an extension of the works in [20,13,15]. Both problems also arise naturally as an extension of [9].

2. Extensions of an arithmetic progression

We use the notation $\mathscr{A} = AP(a,d;n) = \{a+jd\colon\ 0 \leqslant j \leqslant n-1\}$. For the purpose of determining $\mu(\mathscr{A})$, it is no loss of generality to assume that $\gcd(a,d)=1$. In this section, we consider the case when M contains $\mathscr{A}$.

We first consider $\mu(M)$ when M is an 1-element extension of $AP(a,d;n)$. In order for such a set to satisfy $\mu(M) = \mu(\mathscr{A})$, it is sufficient to show that $\mu(M) \geqslant \mu(\mathscr{A})$ since the reverse inequality is obvious as $\mathscr{A} \subset M$. In particular, this gives results on 3-element sets.

Theorem 1. *Let $\mathscr{A} = \{a, a+d, a+2d, \ldots, a+(n-1)d\}$, where $\gcd(a,d)=1$ and d is even. If $b \notin \mathscr{A}$, then*

$$\mu(\mathscr{A} \cup \{b\}) \begin{cases} = \frac{1}{2} & \text{if } b \text{ is odd;} \\ \geqslant \frac{b}{2(a+(n-1)d+b)} & \text{if } b \text{ is even.} \end{cases}$$

Proof. Write $M = \mathscr{A} \cup \{b\}$. If b is odd, all elements of M are odd, and the assertion is obvious since $\{1,3,5,\ldots\}$ is an M-set with density $\frac{1}{2}$.

For even b, write $m' = a+(n-1)d+b$. Observe that m' is odd. Choose $x = \frac{m'-1}{2}$. Then $\gcd(x,m')=1$, and for such x, an easy computation shows that $(a+(i-1)d)x \equiv \frac{m'}{2} - \frac{a+(i-1)d}{2} \pmod{m'}$

for $1 \leqslant i \leqslant n$. By (1),

$$\mu(M) \geqslant \frac{1}{m'} \min_{1\leqslant i\leqslant n}\left(\frac{m'}{2} - \frac{a+(i-1)d}{2}\right) = \frac{b}{2(a+(n-1)d+b)}.$$

This completes the proof. □

Remark 1. The lower bound in Theorem 1 for even b is strong only when b is large.

Conjecture 1.

$$\kappa\big(\mathscr{A} \cup \{b\}\big) = \frac{b}{2(a+(n-1)d+b)}$$
$$\text{for all } b \geqslant \frac{1}{2(n-1)}\big(2a+(n-1)(d-2)\big)\big(a+(n-1)d\big),\ b \text{ even}.$$

The lower bound in the second case of Theorem 1 can be improved for some choices of b. Moreover, we can obtain $\kappa(M)$ in some cases when b and d are both even.

Theorem 2. *Let $\mathscr{A} = \{a, a+d, a+2d, \ldots, a+(n-1)d\}$, where $\gcd(a,d)=1$ and d is even. Let $m = 2a+(n-1)d$.*

(a) *If $d \equiv 2 \pmod 4$, set*

$$S_i := \left\{\frac{d}{2} + a + jd + im\colon\ 0 \leqslant j \leqslant n-2\right\}, \qquad B := \bigcup_{i\geqslant 0} S_i.$$

(b) *If $d \equiv 0 \pmod 4$, set*

$$T_i := \begin{cases} \{\frac{d}{2}n + 2a + jd + im\colon\ 0 \leqslant j \leqslant n-2\} & \text{if } n \text{ is odd;} \\ \{\frac{d}{2}(n-1) + 2a + jd + im\colon\ 0 \leqslant j \leqslant n-1\} & \text{if } n \text{ is even,} \end{cases}$$

and

$$B := \bigcup_{i\geqslant 0} T_i \bigcup \left\{\frac{2k+1}{2}d\colon\ 0 \leqslant k \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor\right\}.$$

Then, in each case,

$$\mu(\mathscr{A} \cup B) \geqslant \frac{m-2(n-1)}{2m}.$$

Proof. We note that m is even, and that $\gcd(d,m)=2$. We use (1) to obtain a lower bound for $\mu(\mathscr{A} \cup B)$.

(a) Suppose $d \equiv 2 \pmod 4$. Choose x such that $\frac{d}{2}x \equiv 1 \pmod m$. Then x is odd, and so

$$ax = \frac{m-(n-1)d}{2}x \equiv \frac{m}{2} - (n-1) \pmod m. \tag{3}$$

Hence, for $0 \leqslant j \leqslant n-1$,

$$(a+jd)x \equiv ax+2j \equiv \frac{m}{2}+\big(2j-(n-1)\big) \pmod m, \tag{4}$$

and for each $b \in B$,

$$bx \equiv \left(\frac{d}{2}+a+jd\right)x \equiv \frac{m}{2}+\big(2j-(n-1)\big)+1 \pmod m. \tag{5}$$

From (3)–(5), we have

$$\mu(\mathscr{A} \cup B) \geqslant \frac{1}{m}\left(\frac{m}{2}-(n-1)\right).$$

(b) Suppose $d \equiv 0 \pmod 4$. Then $m \equiv 2 \pmod 4$ and $\gcd(\frac{d}{2}, m) = 2$. Choose x such that $\frac{d}{2}x \equiv \frac{m}{2}+1 \pmod m$. Replacing x by $x+\frac{m}{2}$ if necessary, we may assume x and n are of the same parity. Then, as in (3) and (4), we have

$$ax = \frac{m-(n-1)d}{2}x \equiv (x-n)\frac{m}{2}+\frac{m}{2}-(n-1) \equiv \frac{m}{2}-(n-1) \pmod m, \tag{6}$$

and for $0 \leqslant j \leqslant n-1$,

$$(2a+jd)x \equiv 2j-2(n-1) \pmod m. \tag{7}$$

If $b = \frac{2k+1}{2}d \in B$, then $bx = (k+\frac{1}{2})dx \equiv \frac{m}{2}+(2k+1) \pmod m$. If $b \in T_i$ for some $i \geqslant 0$ and n is odd, then $\frac{d}{2}nx \equiv \frac{m}{2}+n \pmod m$. If $b \in T_i$ for some $i \geqslant 0$ and n is even, then $\frac{d}{2}(n-1)x \equiv \frac{m}{2}+(n-1) \pmod m$. Using (7), if $b \in B$,

$$bx \equiv \frac{m}{2}+(2k+1) \pmod m \quad \text{with } 0 \leqslant k \leqslant \left\lfloor \frac{n-2}{2} \right\rfloor, \tag{8}$$

or

$$bx \equiv \frac{m}{2}-(n-2)+2j \pmod m \quad \text{with } 0 \leqslant j \leqslant n-2, \tag{9}$$

or

$$bx \equiv \frac{m}{2}-(n-1)+2j \pmod m \quad \text{with } 0 \leqslant j \leqslant n-1. \tag{10}$$

From (8)–(10), we have

$$\mu(\mathscr{A} \cup B) \geqslant \frac{1}{m}\left(\frac{m}{2}-(n-1)\right). \qquad \square$$

Remark 2. If $d \equiv 0 \pmod 4$ and n is odd, we have a slightly better lower bound given by $\mu(\mathscr{A} \cup B) \geqslant \frac{m-2(n-2)}{2m}$ from the proof of Theorem 2.

Remark 3. It is easy to see that $\mu(M) = \frac{1}{2}$ if and only if each element of M is an odd integer or $|M| = 1$. Sufficiency of the condition is clear. For necessity, assume m_1, m_2 are integers in M of opposite parity. Then $\mu(M) \leqslant \mu(\{a, b\}) = \frac{m_1+m_2-1}{2(m_1+m_2)} < \frac{1}{2}$. Since $\mathscr{A}$ consists of only odd integers in this case, $\mu(\mathscr{A} \cup B) = \frac{1}{2}$ whenever each element in B is an odd integer. However in Theorem 2, all elements of B are even integers, and so we only have

$$\frac{1}{2} - \frac{n-1}{2a+(n-1)d} \leqslant \mu(\mathscr{A} \cup B) \leqslant \mu\bigl(\{a+(n-1)d, b_0\}\bigr) = \frac{1}{2} - \frac{1}{2(a+(n-1)d+b_0)},$$

where b_0 is the largest integer in B.

Theorem 3. *Let* $\mathscr{A} = \{a, a+d, a+2d, \ldots, a+(n-1)d\}$, *where* $\gcd(a, d) = 1$ *and* d *is odd. Let* $m = 2a + (n-1)d$. *Set*

$$S_i := \{im + a + jd\colon\ 0 \leqslant j \leqslant n-1\}, \qquad \mathcal{S} := \bigcup_{i \geqslant 1} S_i.$$

Then

$$\mu(\mathscr{A} \cup B) = \kappa(\mathscr{A} \cup B) = \kappa(\mathscr{A}) = \mu(\mathscr{A})$$

for any $B \subseteq \mathcal{S}$. *Conversely,* $\kappa(\mathscr{A} \cup B) = \kappa(\mathscr{A})$ *implies* $B \subseteq \mathcal{S}$.

Proof. Suppose $B \subseteq \mathcal{S}$. In order to show that $\mu(\mathscr{A} \cup B) = \mu(\mathscr{A})$, it suffices to show that $\mu(\mathscr{A} \cup B) \geqslant \mu(\mathscr{A})$. We use (1) to show that $\mu(\mathscr{A})$ is a lower bound for $\mu(\mathscr{A} \cup B)$. Let $m = 2a + (n-1)d$. Then $\gcd(d, m) = 1$. Choose x such that $dx \equiv 1 \pmod m$, and write $dx = 1 + mq$ with $q \in \mathbb{Z}$. Since x and $(n-1)q$ are of opposite parity,

$$ax = \frac{m-(n-1)d}{2}x = \frac{m\{x-(n-1)q\}-(n-1)}{2} \equiv \frac{m-(n-1)}{2} \pmod m.$$

Hence

$$(a+jd)x \equiv \frac{m-(n-1)}{2} + j \pmod m$$

for $0 \leqslant j \leqslant n-1$. Since each $b \in B$ is congruent modulo m to some element of $\mathscr{A}$,

$$\mu(\mathscr{A} \cup B) \geqslant \frac{1}{m} \min_{0 \leqslant j \leqslant n-1}\left(\frac{m-(n-1)}{2} + j\right) = \frac{m-(n-1)}{2m} = \kappa(\mathscr{A}) = \mu(\mathscr{A}).$$

Conversely, suppose $\kappa(\mathscr{A} \cup B) = \kappa(\mathscr{A}) = \mu(\mathscr{A}) = \frac{m-(n-1)}{2m}$. From (1), there exists x_0 such that $1 \leqslant x_0 \leqslant \frac{m}{2}$, $\gcd(x_0, m) = 1$ and $\frac{1}{m}\min_{c \in \mathscr{A} \cup B} |cx_0|_m = \frac{m-(n-1)}{2m}$. In particular, $cx_0 \pmod m \in \mathscr{I} := [\frac{m-(n-1)}{2}, \frac{m+(n-1)}{2}]$ whenever $c \in \mathscr{A} \cup B$. As $\gcd(dx_0, m) = 1$, the elements $\{(a+jd)x_0 \pmod m\colon\ 0 \leqslant j \leqslant n-1\}$ are distinct, and since these n integers all lie in the interval $\mathscr{I}$ consisting of n consecutive integers, $(a+(j+1)d)x_0 - (a+jd)x_0 = dx_0 \equiv \pm 1 \pmod m$. In either case, $\{(a+jd)x_0 \pmod m\colon\ 0 \leqslant j \leqslant n-1\} = \mathscr{I}$. But then $bx_0 \equiv \pm(a+jd)x_0 \pmod m$ for each $b \in B$, so that $b \equiv \pm(a+jd) \pmod m$ for some j with $0 \leqslant j \leqslant n-1$. This completes the proof. □

Remark 4. Observe that $\kappa(\mathscr{A} \cup B) = \kappa(\mathscr{A})$ implies $\mu(\mathscr{A} \cup B) \leqslant \mu(\mathscr{A}) = \kappa(\mathscr{A}) = \kappa(\mathscr{A} \cup B) \leqslant \mu(\mathscr{A} \cup B)$. Hence the converse part of Theorem 3 implies $\mu(\mathscr{A} \cup B) = \mu(\mathscr{A})$ for precisely the subsets B of $\mathcal{S}$ for which $\kappa(\mathscr{A} \cup B) = \mu(\mathscr{A} \cup B)$.

3. Gaps in an arithmetic progression

Exact calculation of $\mu(M)$ when M is of the form $[1,n]\setminus[a,b]$, was a result of the combined efforts of Wu and Lin in [20], Lam and Lin in [13], and Liu and Zhu in [15]. We attempt to extend their results by considering those sets M obtained by removing consecutive terms from any arithmetic progression. More specifically, let $\mathscr{A}=AP(a,d;n)=\{a+jd\colon 0\leqslant j\leqslant n-1\}$ with $\gcd(a,d)=1$, denote an n-term arithmetic progression. We consider cases where M is obtained from $\mathscr{A}$ by removing some k consecutive terms of $\mathscr{A}$. We obtain bounds in all cases. We conjecture these give the exact values of $\kappa(M)$, except in one case where we obtain $\mu(M)$. Since the case of even d leads to only odd terms and consequently to a density $\frac{1}{2}$, we only consider the case of odd d.

Lemma 1. *Let $a,d,n\in\mathbb{N}$. Let $m=2a+(n-1)d$. For each $r\geqslant 0$, define*

$$A_r:=\{mr+s\colon 1\leqslant s\leqslant a+(n-1)d\},\qquad B_r:=\{mr+s\colon 1\leqslant s\leqslant a\}+a+(n-1)d.$$

Then the collection $\{A_0,B_0,A_1,B_1,A_2,B_2,\ldots\}$ partitions $\mathbb{N}$.

Proof. Observe that $A_0=[1,a+(n-1)d]$, $B_0=[a+(n-1)d+1,m]$, and for $r\geqslant 0$, $A_{r+1}=A_r+m$ and $B_{r+1}=B_r+m$. Hence the result. □

Theorem 4. *Let $a,d,n_1,n_2,k\in\mathbb{N}$, with $\gcd(a,d)=1$ and d odd. Let $n=n_1+n_2+k$, $m=2a+(n-1)d$, and $m_1=2a+(n_1-1)d$. Let*

$$M_1=\{a+id\colon 0\leqslant i\leqslant n_1-1\},\qquad M_2=\{a+id\colon n_1+k\leqslant i\leqslant n-1\}.$$

For $r\geqslant 0$, define

$$A_r:=\{m_1r+s\colon 1\leqslant s\leqslant a+(n_1-1)d\},\qquad B_r:=\{m_1r+s\colon 1\leqslant s\leqslant a\}+a+(n_1-1)d.$$

If $n_1=|M_1|\geqslant|M_2|=n_2$, then

$$\mu(M_1\cup M_2)\begin{cases}\geqslant\frac{1}{2}-\frac{(n_1-1)dr+n_2+s-1}{2m} & \text{if } n_1+k\in A_r \text{ and } s\geqslant n_1-n_2;\\ =\frac{2a+(n_1-1)(d-1)}{2(2a+(n_1-1)d)} & \text{if } n_1+k\in A_r \text{ and } s<n_1-n_2;\\ \geqslant\frac{1}{2}-\frac{(n_1-1)(d(r+1)+1)+a-s}{2(m+(n_1-n_2)d)} & \text{if } n_1+k\in B_r,\end{cases}$$

where $s=n_1+k-m_1r$ if $n_1+k\in A_r$ and $s=n_1+k-m_1(r+1)+a$ if $n_1+k\in B_r$.

Proof. Throughout the proof, we use (1) to find a lower bound for $\kappa(M_1\cup M_2)$. This will also serve as a lower bound for $\mu(M_1\cup M_2)$. In case $n_1+k\in A_r$ for some r and for some $s<n_1-n_2$, this lower bound coincides with the upper bound $\mu(M_1)$, thus providing an exact value for $\mu(M_1\cup M_2)$.

Case I. $(n_1+k\in\bigcup_{r\geqslant 0}A_r)$.

We write $n_1+k=m_1r+s$ for some $r\geqslant 0$ and $1\leqslant s\leqslant a+(n_1-1)d$.

Subcase 1. Suppose $s\geqslant n_1-n_2$. Observe that $\gcd(d,m)=1$. Choose x such that $dx\equiv dr+1 \pmod{m}$. Since $2a\equiv-(n_1+n_2+k-1)d \pmod{m}$ and $n_1+k=[m-(n_2+k)d]r+s\equiv-(n_2+k)dr+s \pmod{m}$, we have

$$\begin{aligned}2ax &\equiv -(n_1+n_2+k-1)dx \equiv -(n_1+n_2+k-1)(dr+1) \pmod m\\ &\equiv -\big[(n_1-1)dr+(n_2+k)dr+(n_1+k)+(n_2-1)\big]\\ &\equiv -\big[(n_1-1)dr+s+(n_2-1)\big] \pmod m.\end{aligned}$$

Since $n_1+k \in A_r$, we can write

$$m = 2a+\big\{\big(2a+(n_1-1)d\big)r+s+n_2-1\big\}d = 2a(dr+1)+d\big\{(n_1-1)dr+s+(n_2-1)\big\}.$$

Hence

$$dax - d\frac{m-\{(n_1-1)dr+n_2+s-1\}}{2} = dax - \frac{d-1}{2}m - a(dr+1) \equiv 0 \pmod m,$$

so that

$$ax \equiv \frac{m}{2} - \frac{(n_1-1)dr+n_2+s-1}{2} \pmod m. \tag{11}$$

Set $\ell := (n_1-1)dr+n_2+s-1$, and $\mathscr{I} := [\frac{m}{2}-\frac{\ell}{2}, \frac{m}{2}+\frac{\ell}{2}]$.

Then

$$\big(a+(n_1-1)d\big)x \equiv \frac{m}{2}-\frac{\ell}{2}+(n_1-1)dx \equiv \frac{m}{2}+\frac{\ell}{2}-\big(s-(n_1-n_2)\big) \pmod m.$$

Thus $m_i x \pmod m \in \mathscr{I}$ for each $m_i \in M_1 \cup M_2$ since $n_2 \leqslant n_1$ implies $\{m-m_i\colon m_i \in M_2\} \subseteq M_1$. This proves the first part of the assertion, that

$$\mu(M) \geqslant \frac{m-\{(n_1-1)dr+n_2+s-1\}}{2m}.$$

Subcase 2. Suppose $s < n_1-n_2$. Let $m' = m+(n_1-n_2-s)d$. Since $n_1+k \in A_r$, we can write $m' = 2a+((n_1-1)+m_1r)d = (dr+1)m_1$. Thus $\gcd(d,m')=1$. Choose x such that $dx \equiv dr+1 \pmod{m'}$. Since $(n_1+k-s)(dr+1) = m_1r(dr+1) = m'r$, we have

$$\begin{aligned}2ax &\equiv -(2n_1+k-s-1)dx \equiv -\big(n_1+k-s+(n_1-1)\big)(dr+1)\\ &\equiv -(n_1-1)(dr+1) \pmod{m'}.\end{aligned}$$

Since

$$m' - d(n_1-1)(dr+1) = (dr+1)\big(m_1-(n_1-1)d\big) = 2a(dr+1),$$

we have

$$dax - d\frac{m'-(n_1-1)(dr+1)}{2} = dax - \frac{d-1}{2}m' - a(dr+1) \equiv 0 \pmod{m'},$$

and

$$ax \equiv \frac{m'}{2} - \frac{(n_1-1)(dr+1)}{2} \pmod{m'}. \tag{12}$$

Set $\ell := (n_1-1)(dr+1)$, and $\mathscr{I} := [\frac{m'}{2}-\frac{\ell}{2}, \frac{m'}{2}+\frac{\ell}{2}]$.

Then

$$\big(a+(n_1-1)d\big)x \equiv \frac{m'}{2}-\frac{\ell}{2}+(n_1-1)dx \equiv \frac{m'}{2}+\frac{\ell}{2} \pmod{m'}.$$

Thus $m_i x \pmod{m'} \in \mathscr{I}$ for each $m_i \in M_1 \cup M_2$ since $n_2 \leqslant n_1$ implies $\{m'-m_i\colon m_i \in M_2\} \subseteq M_1$. Thus

$$\mu(M) \geqslant \frac{m'-(n_1-1)(dr+1)}{2m'} = \frac{m_1-(n_1-1)}{2m_1},$$

since $m' = m_1(dr+1)$. On the other hand, $\mu(M) \leqslant \mu(M_1) = \frac{m_1-(n_1-1)}{2m_1}$, so the equality holds for this case.

Case II. $(n_1+k \in \bigcup_{r\geqslant 0} B_r)$.

We write $n_1+k = m_1 r + s + a + (n_1-1)d$ for some $r \geqslant 0$ and $1 \leqslant s \leqslant a$. Let $m_0 = m + (n_1-n_2)d$. Then $\gcd(d, m_0) = 1$. Choose x such that $dx \equiv d(r+1)+1 \pmod{m_0}$. Since $2a \equiv -(2n_1+k-1)d \pmod{m_0}$ and $n_1+k = (m_0-(n_1+k)d)(r+1) - a + s \equiv -(n_1+k)d(r+1) - a + s \pmod{m_0}$ or $(n_1+k)(d(r+1)+1) \equiv s-a \pmod{m_0}$, we have

$$\begin{aligned} 2ax &\equiv -(2n_1+k-1)dx \equiv -\big[(n_1-1)+(n_1+k)\big]\big[d(r+1)+1\big] \pmod{m_0} \\ &\equiv -(n_1-1)\big(d(r+1)+1\big)+a-s \pmod{m_0}. \end{aligned}$$

Since $n_1+k \in B_r$, we can write

$$n_1+k = \big(2a+(n_1-1)d\big)r + s + a + (n_1-1)d = \big(2a+(n_1-1)d\big)(r+1) - a + s,$$

and so

$$m_0 = 2a+(n_1-1)d+(n_1+k)d = \big(2a+(n_1-1)d\big)\big(d(r+1)+1\big) - d(a-s).$$

Hence

$$\begin{aligned} dax - d\frac{m_0-[(n_1-1)(d(r+1)+1)-(a-s)]}{2} &= dax - \frac{d-1}{2}m_0 - a\big(d(r+1)+1\big) \\ &\equiv 0 \pmod{m_0}, \end{aligned}$$

so that

$$ax \equiv \frac{m_0}{2} - \frac{1}{2}(n_1-1)\big(d(r+1)+1\big) + \frac{a-s}{2} \pmod{m_0}. \tag{13}$$

Set $\ell := (n_1-1)(d(r+1)+1)+a-s$, and $\mathscr{I} := [\frac{m_0}{2}-\frac{\ell}{2}, \frac{m_0}{2}+\frac{\ell}{2}]$.

Then

$$\big(a+(n_1-1)d\big)x \equiv \frac{m_0}{2}-\frac{\ell}{2}+(a-s)+(n_1-1)dx \equiv \frac{m_0}{2}+\frac{\ell}{2} \pmod{m_0}.$$

Since $n_2 \leqslant n_1$ implies $\{m_0 - m_i\colon\ m_i \in M_2\} \subseteq M_1$, we have the second part of the assertion, that

$$\mu(M) \geqslant \frac{m_0 - \{(n_1-1)(d(r+1)+1) + a - s\}}{2m_0}.$$

This completes the proof of the theorem. □

Lemma 2. *Let $a, d, n \in \mathbb{N}$. Let $m = 2a + (n-1)d$. For each $r \geqslant 0$, define*

$$C_r := \{mr + s\colon\ 1 \leqslant s \leqslant (n-1)d\}, \qquad D_r := \{mr + s\colon\ 1 \leqslant s \leqslant 2a\} + (n-1)d.$$

Then the collection $\{C_0, D_0, C_1, D_1, C_2, D_2, \ldots\}$ partitions $\mathbb{N}$.

Proof. Observe that $C_0 = [1, (n-1)d]$, $D_0 = [(n-1)d + 1, m]$, and for $r \geqslant 0$, $C_{r+1} = C_r + m$ and $D_{r+1} = D_r + m$. Hence the result. □

Theorem 5. *Let $a, d, n_1, n_2, k \in \mathbb{N}$, with $\gcd(a, d) = 1$ and d odd. Let $n = n_1 + n_2 + k$, $m = 2a + (n-1)d$, and $m_2 = 2a + (n_2 - 1)d$. Let*

$$M_1 = \{a + id\colon\ 0 \leqslant i \leqslant n_1 - 1\}, \qquad M_2 = \{a + id\colon\ n_1 + k \leqslant i \leqslant n - 1\}.$$

For $r \geqslant 0$, define

$$C_r := \{m_2 r + s\colon\ 1 \leqslant s \leqslant (n_2-1)d\}, \qquad D_r := \{m_2 r + s\colon\ 1 \leqslant s \leqslant 2a\} + (n_2-1)d.$$

If $n_1 = |M_1| < |M_2| = n_2$ and $N = 2a - (n_2 - n_1)(2d(r+1)+1)$, then

$$\mu(M_1 \cup M_2) \geqslant \begin{cases} \frac{1}{2} - \frac{(n_2-1)(dr+1)+s}{2m} & \text{if } n_1 + k \in C_r; \\ \frac{1}{2} - \frac{(n_2-1)(d(r+1)+1)+s}{2m} & \text{if } n_1 + k \in D_r \text{ and } 1 \leqslant s \leqslant \lfloor \frac{N}{2} \rfloor; \\ \frac{1}{2} - \frac{(n_2-1)(d(r+1)+1)+N-s}{2(m+(n_1-n_2)d)} & \text{if } n_1 + k \in D_r \text{ and } \lceil \frac{N}{2} \rceil \leqslant s \leqslant N; \\ \frac{1}{2} - \frac{(n_2-1)(2d(r+1)+1)}{2(m+(n_1+k)d)} & \text{if } n_1 + k \in D_r \text{ and } s \geqslant N, \end{cases}$$

where $s = n_1 + k - m_2 r$ if $n_1 + k \in C_r$ and $s = n_1 + k - m_2(r+1) + 2a$ if $n_1 + k \in D_r$.

Proof. Throughout the proof, we use (1) to find a lower bound for $\kappa(M_1 \cup M_2)$. This will also serve as a lower bound for $\mu(M_1 \cup M_2)$.

Case I. *($n_1 + k \in \bigcup_{r \geqslant 0} C_r$).*

Note that $\gcd(d, m) = 1$. Choose x such that $dx \equiv dr + 1 \pmod{m}$. Since $2a \equiv -(n_1 + n_2 + k - 1)d \pmod{m}$ and $n_1 + k = (m - (n_1+k)d)r + s \equiv -(n_1+k)dr + s \pmod{m}$ or $(n_1+k)(dr+1) \equiv s \pmod{m}$, we have

$$\begin{aligned} 2ax &\equiv -(n_1 + n_2 + k - 1)dx \equiv -(n_1 + n_2 + k - 1)(dr+1) \\ &\equiv -[(n_2 - 1)(dr+1) + s] \pmod{m}. \end{aligned}$$

Since $n_1 + k \in C_r$, we can write

$$m = 2a + (n_2 - 1)d + \{[2a + (n_2 - 1)d]r + s\}d = 2a(dr + 1) + d\{(n_2 - 1)(dr + 1) + s\}.$$

Hence

$$dax - d\frac{m - (n_2 - 1)(dr + 1) - s}{2} = dax - \frac{d - 1}{2}m + a(dr + 1) \equiv 0 \pmod{m},$$

so that

$$ax \equiv \frac{m}{2} - \frac{(n_2 - 1)(dr + 1) + s}{2} \pmod{m}. \tag{14}$$

Set $\ell := (n_2 - 1)(dr + 1) + s$, and $\mathscr{I} := [\frac{m}{2} - \frac{\ell}{2}, \frac{m}{2} + \frac{\ell}{2}]$.

Then

$$\bigl(a + (n_1 + k)d\bigr)x \equiv \frac{m}{2} - \frac{\ell}{2} + (n_1 + k)dx \equiv \frac{m}{2} - \frac{\ell}{2} + s \pmod{m}.$$

Since $a + (n_1 + n_2 + k - 1)d \equiv -a \pmod{m}$, we also have

$$\bigl(a + (n_1 + n_2 + k - 1)d\bigr)x \equiv -ax \equiv \frac{m}{2} + \frac{\ell}{2} \pmod{m}.$$

Thus $m_i x \pmod{m} \in \mathscr{I}$ for each $m_i \in M_1 \cup M_2$ since $n_1 < n_2$ implies $\{m - m_i\colon\ m_i \in M_1\} \subseteq M_2$. This proves the first part of the assertion, that

$$\mu(M) \geqslant \frac{m - \{(n_2 - 1)(dr + 1) + s\}}{2m}.$$

Case II. $(n_1 + k \in \bigcup_{r \geqslant 0} D_r)$.

Subcase 1. Suppose $1 \leqslant s \leqslant \lfloor \frac{N}{2} \rfloor$. Note that $\gcd(d, m) = 1$. Choose x such that $dx \equiv dr + 1 \pmod{m}$, as in Case I. For $n_1 + k \in D_r$, we can use the result in Case I to write $n_1 + k = (m - (n_1 + k)d)r + s + (n_2 - 1)d \equiv -(n_1 + k)dr + s + (n_2 - 1)d \pmod{m}$ or $(n_1 + k)(dr + 1) \equiv s + (n_2 - 1)d \pmod{m}$, and

$$2ax \equiv -(n_1 + n_2 + k - 1)(dr + 1) \equiv -\bigl[(n_2 - 1)\bigl(d(r + 1) + 1\bigr) + s\bigr] \pmod{m}.$$

As in Case I, it follows that

$$ax \equiv \frac{m}{2} - \frac{(n_2 - 1)(d(r + 1) + 1) + s}{2} \pmod{m}. \tag{15}$$

Set $\ell := (n_2 - 1)(d(r + 1) + 1) + s$, and $\mathscr{I} := [\frac{m}{2} - \frac{\ell}{2}, \frac{m}{2} + \frac{\ell}{2}]$.

Then

$$\bigl(a + (n_1 + k)d\bigr)x \equiv \frac{m}{2} - \frac{\ell}{2} + (n_1 + k)dx \equiv \frac{m}{2} - \frac{\ell}{2} + (n_2 - 1)d + s \pmod{m}.$$

Since $a + (n_1 + n_2 + k - 1)d \equiv -a \pmod m$, we also have

$$\big(a + (n_1 + n_2 + k - 1)d\big)x \equiv -ax \equiv \frac{m}{2} + \frac{\ell}{2} \pmod m.$$

Thus $m_i x \pmod m \in \mathscr{I}$ for each $i \in M_1 \cup M_2$ since $n_1 < n_2$ implies $\{m - m_i\colon m_i \in M_1\} \subseteq M_2$. Hence

$$\mu(M) \geqslant \frac{m - \{(n_2 - 1)(d(r + 1) + 1) + s\}}{2m}.$$

Subcase 2. Suppose $\lceil \frac{N}{2} \rceil \leqslant s \leqslant N$. Let $m_0 = m + (n_1 - n_2)d$. Then $\gcd(d, m_0) = 1$. Choose x such that $dx \equiv d(r + 1) + 1 \pmod{m_0}$. Since $2a \equiv -(2n_1 + k - 1)d \pmod{m_0}$ and $n_1 + k = (m_0 - (2n_1 - n_2 + k)d)r + s + (n_2 - 1)d \equiv -2(n_1 + k)dr + (n_2 + k)dr + (n_2 - 1)d + s \pmod{m_0}$ or $(n_1 + k)(2dr + 1) \equiv (n_2 - 1)d(r + 1) + (k + 1)dr + s \pmod{m_0}$, we have

$$\begin{aligned} 2ax &\equiv -(2n_1 + k - 1)\big(d(r + 1) + 1\big) \equiv 2a(r + 1) - (n_1 - 1) - (n_1 + k) \\ &= 2a - (n_1 - 1) + (2a - m_2)r - s - (n_2 - 1)d \\ &= 2a - \big\{(n_1 - 1) + (n_2 - 1)d(r + 1) + s\big\} \pmod{m_0}. \end{aligned}$$

Hence

$$ax \equiv \frac{m_0}{2} + \frac{2a - ((n_1 - 1) + (n_2 - 1)d(r + 1) + s)}{2} \pmod{m_0}. \tag{16}$$

Set $\ell := (n_2 - 1)(d(r + 1) + 1) + N - s$, and $\mathscr{I} := [\frac{m_0}{2} - \frac{\ell}{2}, \frac{m_0}{2} + \frac{\ell}{2}]$.

Then

$$\big(a + (n_1 + k)d\big)x \equiv -\big(a + (n_1 - 1)d\big)x \equiv \frac{m_0}{2} - \frac{\ell}{2} \pmod{m_0},$$

$$\big(a + (n_1 + n_2 + k - 1)d\big)x \equiv \frac{m_0}{2} - \frac{\ell}{2} + (n_2 - 1)\big(d(r + 1) + 1\big) \pmod{m_0}.$$

Note that $(n_2 - 1)(d(r + 1) + 1) \leqslant \ell$ since $s \leqslant N$. Thus $m_i x \pmod{m_0} \in \mathscr{I}$ for each $m_i \in M_1 \cup M_2$ since $n_1 < n_2$ implies $\{m_0 - m_i\colon m_i \in M_1\} \subseteq M_2$. Thus

$$\mu(M) \geqslant \frac{m_0 - \{(n_2 - 1)(d(r + 1) + 1) + N - s\}}{2m_0}.$$

Subcase 3. Suppose $s \geqslant N$. Let $m' = m + (n_1 + k)d$. Then $\gcd(d, m') = 1$. Choose x such that $dx \equiv 2d(r + 1) + 1 \pmod{m'}$. Since $2a \equiv -[2(n_1 + k) + n_2 - 1]d \pmod{m'}$ and $n_1 + k = [m' - 2(n_1 + k)d]r + s + (n_2 - 1)d \equiv -2(n_1 + k)dr + (n_2 - 1)d + s \pmod{m'}$ or $(n_1 + k)(2dr + 1) \equiv (n_2 - 1)d + s \pmod{m'}$, we have

$$2ax \equiv -2(n_1 + k)dx - (n_2 - 1)dx \equiv 4a - 2s - (n_2 - 1)\big(2d(r + 1) + 1\big) \pmod{m'},$$

and

$$ax \equiv \frac{m'}{2} - \frac{(n_2 - 1)(2d(r + 1) + 1) + 4a - 2s}{2} \pmod{m'}. \tag{17}$$

Set $\ell := (n_2 - 1)(2d(r + 1) + 1)$, and $\mathscr{I} := [\frac{m'}{2} - \frac{\ell}{2}, \frac{m'}{2} + \frac{\ell}{2}]$.

Then

$$\big(a+(n_1+k)d\big)x \equiv \frac{m'}{2}-\frac{\ell}{2} \pmod{m'},$$
$$\big(a+(n_1+n_2+k-1)d\big)x \equiv \frac{m'}{2}+\frac{\ell}{2} \pmod{m'},$$

and

$$\big(a+(n_1-1)d\big)x \equiv \frac{m'}{2}-\frac{\ell}{2}+2a-s+(n_1-1)\big(2d(r+1)+1\big) \pmod{m'}.$$

Note that $2a-s+(n_1-1)(2d(r+1)+1) \leqslant \ell$ since $s \geqslant N$. Thus $m_i x \pmod{m'} \in \mathscr{I}$ for each $m_i \in M_1 \cup M_2$ since $n_1 < n_2$ implies $\{m'-m_i\colon m_i \in M_1\} \subseteq M_2$. Thus

$$\mu(M) \geqslant \frac{m'-\{(n_2-1)(2d(r+1)+1)\}}{2m'}.$$

This completes the proof. □

Remark 5. We note that both $\mu(M_1)$ and $\mu(M_2)$ serve as upper bounds for $\mu(M_1 \cup M_2)$ while $\mu(\mathscr{A})$ serves as a lower bound. The second case of Theorem 4 provides an instance of one of the upper bounds being achieved.

Conjecture 2. *The lower bounds for $\mu(M_1 \cup M_2)$ in Theorems 4 and 5 equal $\kappa(M_1 \cup M_2)$.*

Closing Remarks. The sets we have considered here are either contained in or contain an arithmetic progression. The computation of $\kappa(\cdot)$, that is intrinsic to the Lonely Runner Conjecture, provides a lower bound for $\mu(\cdot)$, by (1). The computation of $\kappa(\cdot)$ involves the correct combination of x and m in (1), or equivalently by a result of Haralambis in [11], to maximizing the term over only those values of m that are sums of two distinct integers in M and x is a positive integer not exceeding $m/2$. Our choice of m and x in each result has been based on a programme to compute $\kappa(\cdot)$ via this variation of (1), and we have attempted to formulate and prove our results based on these. Unfortunately there are still a large number of values of m to compare, and this makes the exact computation of $\kappa(\cdot)$ a computationally unrewarding task even by this variation. Therefore we have provided the correct choice of m and x in every case, but not provided a proof for this while computing $\kappa(\cdot)$. Whenever possible, we have also determined $\mu(\cdot)$ by proving that the lower bound $\kappa(\cdot)$ equals an upper bound provided by μ for an arithmetic progression.

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