A comparison of dispersion and Markoff constants*

by

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0 Introduction

Let $\{x_n\}$ be a sequence of numbers, $0 \leq x_n \leq 1$. In [3], H. Niederreiter introduced a measure of denseness of such a sequence as follows: For each $N \geq 1$, let

$$d_N = \sup_{0 \leq x \leq 1} \min_{1 \leq n \leq N} |x - x_n|$$

and define

$$D(\{x_n\}) = \limsup_{N \to \infty} N\, d_N.$$

In particular, for irrational α, the *dispersion constant* $D(\alpha)$ is defined by $D(\{n\alpha \bmod 1\})$. It is well known that, for irrational α, the *Markoff constant* $M(\alpha)$ is defined by

$$M(\alpha)^{-1} = \liminf_{n \to \infty} n||n\alpha||,$$

where $||x||$ denotes the distance from x to the nearest integer.

In [3], Niederreiter asks if $M(\alpha) < M(\beta)$ implies $D(\alpha) < D(\beta)$. V. Drobot [1] has shown this to be false by producing a counterexample of two quadratic irrationals, both with continued fraction expansions with period length nine. In this paper, we classfy some infinte families of apirs (α, β) of irrational numbers that satisfy $M(\alpha) < M(\beta)$ and $D(\alpha) > D(\beta)$.

We first outline the method of V. Drobot [1] to compute $D(\alpha)$ for quadratic irrationals α. If α is a real irrational number with continued fraction expansion $\alpha = [a_0; a_1, a_2, \ldots]$, let

$$\lambda_i = [0; a_i, a_{i-1}, \ldots, a_1], \quad \Lambda_i = [a_{i+1}; a_{i+2}, \ldots], \quad M_i = \lambda_i + \Lambda_i.$$

We define

$$\psi_i(x) = \frac{1}{M_i}[-x^2 + (\Lambda_i - \lambda_i - 1)x + \Lambda_i(1 + \lambda_i)],$$

$$x_i = \frac{\Lambda_i - \lambda_i - 1}{2}, \quad n_i \text{ is the integer closest to } x_i.$$

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Then Drobot [1] has shown that $D(\alpha) = \limsup_{i\to\infty} \psi_i(n_i)$. In particular, if α has a periodic continued fraction expansion, there are only finitely many choices for i and taking the lim sup reduces to taking the maximum of the values $\psi_i(n_i) = D_i$ of the quadratic polynomial $\psi_i(x)$. In view of [3], Theorem 6, p. 1197, it is no restriction to suppose that this expansion is purely periodic and that $a_0 \geq 1$:

$$\alpha = [\overline{a_0; a_1, \ldots, a_{k-1}}], a_0 \geq 1.$$

We extend the periodic sequence

$$c_0; c_1, c_2, \ldots = a_0; a_1, a_2, \ldots, a_{k-1}, a_0, a_1, \ldots$$

periodically in the other direction as well, that is,

$$\ldots c_{-2}, c_{-1}, c_0; c_1, c_2, \ldots = \ldots a_0, a_1, \ldots, a_{k-1}, a_0; a_1, \ldots, a_{k-1}, a_0, a_1, \ldots$$

Then, with $\overline{\lambda_i} = [0; c_i, c_{i-1}, \ldots]$, $\overline{\Lambda_i} = [c_{i+1}; c_{i+2}, \ldots]$, $\overline{M_i} = \overline{\lambda_i} + \overline{\Lambda_i}$, and similar definitions for $\overline{\psi}_i(x)$, $\overline{x}_i$ and $\overline{n}_i$, one has (see [1], p. 93) $D(\alpha) = \max_{0\leq i\leq k-1} \overline{\psi}_i(n_i)$. In this paper, we only deal with the quadratic irrational case, therefore we may and do, omit the bar in the above notation without causing any confusion.

We use this method to determine $D(\alpha)$ for α that has a purely periodic continued fraction expansion. Let

$$\alpha = \frac{A + \sqrt{D}}{B},$$

where

$$A = p_{k-1}(\alpha) - q_{k-2}(\alpha), \quad B = 2q_{k-1}(\alpha), \quad D = (p_{k-1}(\alpha) + q_{k-2}(\alpha))^2 - 4(-1)^k;$$

here $p_n(\alpha)/q_n(\alpha)$ is the nth convergent to $[a_0; a_1, \ldots, a_{k-1}]$. If we set $\Lambda_i = (a + \sqrt{D})/b$, then $\lambda_i = (-a + \sqrt{D})/b$, so that $M_i = 2\sqrt{D}/b$ and $2x_i = (2a - b)/b$.

In particular, if $n_i = 0$, then

$$M_i D_i = \Lambda_i + \Lambda_i \lambda_i = \frac{a + \sqrt{D}}{b} + \frac{D - a^2}{b^2},$$

so that

$$\begin{aligned}
(A) \quad D_i &= \frac{1}{M_i}\left(\frac{a}{b} + \frac{D - a^2}{b^2}\right) + \frac{1}{2} \\
&= q_{k-1}(\Lambda_i)[(p_{k-1}(\Lambda_i) + q_{k-2}(\Lambda_i))^2 - 4(-1)^k]^{-1/2} \\
&\quad \times \left(\frac{p_{k-1}(\Lambda_i) - q_{k-2}(\Lambda_i)}{2q_{k-1}(\Lambda_i)} + \frac{p_{k-1}(\Lambda_i)q_{k-2}(\Lambda_i) - (-1)^k}{q_{k-1}(\Lambda_i)^2}\right) + \frac{1}{2} \\
&= [(p_{k-1}(\Lambda_i) + q_{k-2}(\Lambda_i))^2 - 4(-1)^k]^{-1/2}\left(\frac{p_{k-1}(\Lambda_i) - q_{k-2}(\Lambda_i)}{2} + p_{k-2}(\Lambda_i)\right) + \frac{1}{2}
\end{aligned}$$

If $n_i = 1$, then

$$M_i D_i = 2x_i - 1 + \Lambda_i + \Lambda_i \lambda_i = 2\frac{a - b}{b} + \frac{a + \sqrt{D}}{b} + \frac{D - a^2}{b^2},$$

so that

$$
\begin{aligned}
(B)\quad D_i &= [(p_{k-1}(\Lambda_i)+q_{k-2}(\Lambda_i))^2-4(-1)^k]^{-1/2}\\
&\quad\times\left(\frac{p_{k-1}(\Lambda_i)-q_{k-2}(\Lambda_i)}{2}+p_{k-2}(\Lambda_i)+(p_{k-1}(\Lambda_i)-q_{k-2}(\Lambda_i)-2q_{k-1}(\Lambda_i)\right)+\frac{1}{2}\\
&= [(p_{k-1}(\Lambda_i)+q_{k-2}(\Lambda_i))^2-4(-1)^k]^{-1/2}\\
&\quad\times\left(\frac{3}{2}(p_{k-1}(\Lambda_i)-q_{k-2}(\Lambda_i))+p_{k-2}(\Lambda_i)-2q_{k-1}(\Lambda_i)\right)+\frac{1}{2}
\end{aligned}
$$

Thus, if n_i only takes the values 0 and 1 for $i=0,\dots,k-1$, then $D(\alpha)$ is computed by taking the maximum of the relevant expressions given by equations (A), (B).

For each $k\geq 1$, let $\alpha^{(k)}=[\overline{c_0;c_1,c_1,\dots,c_1}]$, with k occurrences of c_1 and where $c_0>c_1$. For each $k\geq 1$, we set

$$\alpha^{(k)}=\Lambda^{(k)}=[\overline{c_0;c_1,c_1,\dots,c_1}],\quad \lambda^{(k)}=[0;\overline{c_1,c_1,\dots,c_1,c_0}]$$

with k occurrences of c_1.

If $\Lambda^{(k)}=(a_k+\sqrt{d_k})/b_k$, then $\lambda^{(k)}=(-a_k+\sqrt{d_k})/b_k$, so that

$$M^{(k)}=\Lambda^{(k)}+\lambda^{(k)}=2\frac{\sqrt{d_k}}{b_k}. \tag{1}$$

In particular, for a fixed k, the numerators and denominators $p_\ell^{(k)}$ and $q_\ell^{(k)}$ of $\alpha^{(k)}$ satisfy

$$
\begin{aligned}
&p_{-1}^{(k)}=1,\quad q_{-1}^{(k)}=0,\quad p_0^{(k)}=c_0,\quad q_0^{(k)}=1,\\
&p_\ell^{(k)}=c_1p_{\ell-1}^{(k)}+p_{\ell-2}^{(k)},\quad q_\ell^{(k)}=c_1q_{\ell-1}^{(k)}+q_{\ell-2}^{(k)},\quad 1\leq\ell\leq k.
\end{aligned} \tag{2}
$$

Since these are second order linear recurrence equations with constant coefficients, it follows from well known facts (see [4], pp. 121-122, for instance) that

$$(r_2-r_1)p_\ell^{(k)}=c_0(r_2^{\ell+1}-r_1^{\ell+1})+r_2^\ell-r_1^\ell,\quad (r_2-r_1)q_\ell^{(k)}=r_2^{\ell+1}-r_1^{\ell+1},\quad \ell=0,1,\dots,k,$$

where r_1,r_2 satisfy the equation $r^2-c_1r-1=0$.

Since $a_k=p_k^{(k)}-q_{k-1}^{(k)}$, $b_k=2q_k^{(k)}$, and $d_k=(p_k^{(k)}+q_{k-1}^{(k)})^2+4(-1)^k$, it follows that

$$M_k(\alpha^{(k)})=\frac{\sqrt{\{p_k^{(k)}+q_{k-1}^{(k)}\}^2+4(-1)^k}}{q_k^{(k)}},$$

so that

$$
\begin{aligned}
M_k(\alpha^{(k)})^2 &= c_0^2+4(r_2^{k+1}-r_1^{k+1})^{-2}\left\{c_0(r_2^k-r_1^k)(r_2^{k+1}-r_1^{k+1})+(r_2^k-r_1^k)^2\right.\\
&\quad\left.+(-1)^k(r_2-r_1)\right\}\\
&= c_0^2+4(r_2^{k+1}-r_1^{k+1})^{-2}\left\{c_0(r_2^{2k+1}+r_1^{2k+1}+(-1)^kc_1)+(r_2^k-r_1^k)^2\right.\\
&\quad\left.+(-1)^k(c_1^2+4)\right\}.
\end{aligned} \tag{3}
$$

With the previous notation, we may now write $\alpha^{(k)} = \Lambda^{(k)} = \Lambda_k(\alpha^{(k)})$ and $\lambda^{(k)} = \lambda_k(\alpha^{(k)})$. For an α of the special form $\alpha^{(k)} = [\overline{c_0; c_1, \ldots, c_1}]$, with k occurrences of c_1, we have $M(\alpha^{(k)}) = \max_{0\le i\le k-1} M_i$ (see [2], formula (11), p. 29). Since $c_0 > c_1$, we have

$$\begin{array}{rcl} M_k(\alpha^{(k)}) & = & c_0 + 2[0; \overline{c_1, \ldots, c_1, c_0}] \quad \geq \quad c_1 + 1 + 2[0; \overline{c_1, \ldots, c_1, c_0}] \\ & > & c_1 + 2[0; \overline{c_1, c_0, c_1, \ldots, c_1}] \quad = \quad M_1(\alpha^{(k)}) \geq M_i(\alpha^{(k)}) \ \text{ for } \ i \neq k. \end{array}$$

Thus,

$$M(\alpha^{(k)}) = M_k(\alpha^{(k)}).$$

Using (3), it is easy to see that

$$M([\overline{c_0}]) = \sqrt{c_0^2 + 4}, \quad M([\overline{c_0, c_1}]) = \sqrt{c_0^2 + 4\frac{c_0}{c_1}}, \ \text{ and } \ M([\overline{c_0, c_1, c_1}]) = \sqrt{c_0^2 + 4\frac{c_0c_1 + 1}{c_1^2 + 1}}.$$

We observe that for each $k \geq 1$, $\Lambda^{(k)} - \lambda^{(k)} = c_0$, so that $n^{(k)} = [c_0/2]$, where $n^{(k)}$ is defined to be the integer closest to $(\Lambda^{(k)} - \lambda^{(k)} - 1)/2$.

Lemma 1. *For each $k \geq 1$,*

$$M(\alpha^{(k)}) \left\{4\, D(\alpha^{(k)}) - M(\alpha^{(k)}) - 2\right\} = \begin{cases} 0 & \textit{if } c_0 \textit{ is even;} \\ 1 & \textit{if } c_0 \textit{ is odd.} \end{cases}$$

Proof. We recall that for α with a periodic continued fraction expansion,

$$D(\alpha) = \max_i \psi_i(n_i) = \max_i \frac{1}{\lambda_i + \Lambda_i} \left\{-n_i^2 + (\Lambda_i - \lambda_i - 1)n_i + \Lambda_i(1 + \lambda_i)\right\}.$$

Since n_i is the integer closest to $(\Lambda_i - \lambda_i - 1)/2$, the expression for $\psi_i(n_i)$ is an increasing function of Λ_i, so that $D(\alpha) = \psi_k(n_k)$, where $n_k = n^{(k)}$. Also, since $M_i = \lambda_i + \Lambda_i$, we have

$$\begin{array}{rcl} M(\alpha^{(k)})D(\alpha^{(k)}) & = & -\left(n^{(k)}\right)^2 + (\Lambda^{(k)} - \lambda^{(k)} - 1)n^{(k)} + \Lambda^{(k)}(\lambda^{(k)} + 1) \\ & = & -\left[\frac{c_0}{2}\right]^2 + (c_0 - 1)\left[\frac{c_0}{2}\right] + \frac{d_k - a_k^2}{b_k^2} + \frac{a_k + \sqrt{d_k}}{b_k} \\ & = & -\left[\frac{c_0}{2}\right]^2 + (c_0 - 1)\left[\frac{c_0}{2}\right] + \frac{a_k}{b_k} - \frac{a_k^2}{b_k^2} + \frac{1}{2}M(\alpha^{(k)}) + \frac{1}{4}M(\alpha^{(k)})^2, \end{array}$$

so that

$$\begin{array}{rcl} M(\alpha^{(k)}) \left\{4\, D(\alpha^{(k)}) - M(\alpha^{(k)}) - 2\right\} & = & 4\left\{\frac{a_k}{b_k} - \frac{a_k^2}{b_k^2} + (c_0 - 1)\left[\frac{c_0}{2}\right] - \left[\frac{c_0}{2}\right]^2\right\} \\ & = & 4\left\{\frac{c_0}{2} - \left(\frac{c_0}{2}\right)^2 + (c_0 - 1)\left[\frac{c_0}{2}\right] - \left[\frac{c_0}{2}\right]^2\right\} \\ & = & \begin{cases} 0 & \text{if } c_0 \text{ is even;} \\ 1 & \text{if } c_0 \text{ is odd.} \end{cases} \end{array}$$

□

1 A condition for $M(\alpha) < M(\beta)$ and $D(\alpha) > D(\beta)$

We recall a result of Drobot [1]: $M(\alpha) \le 4\,D(\alpha) - 2 \le M(\alpha) + M(\alpha)^{-1}$ for any real α. Thus, with each real α, we may associate a constant $k(\alpha) = k \in [0,1]$ such that $4\,D(\alpha) - 2 = M(\alpha) + k\,M(\alpha)^{-1}$.

Thus, we may define

$$k(\alpha) = M(\alpha)(4\,D(\alpha) - M(\alpha) - 2). \tag{4}$$

Suppose that $M(\alpha) < M(\beta)$ but $D(\alpha) > D(\beta)$. Now,

$$D(\alpha) > D(\beta) \Leftrightarrow 4\,D(\alpha) - 2 > 4\,D(\beta) - 2 \Leftrightarrow M(\alpha) + k(\alpha)\,M(\alpha)^{-1} > M(\beta) + k(\beta)\,M(\beta)^{-1},$$

for some $k(\alpha), k(\beta) \in [0,1]$. If we can find α such that $k(\alpha) = 1$ and β such that $k(\beta) = 0$, then this reduces to

$$M(\alpha) < M(\beta) < M(\alpha) + M(\alpha)^{-1} \Leftrightarrow M(\alpha)^2 < M(\beta)^2 < M(\alpha)^2 + M(\alpha)^{-2} + 2 \tag{5}$$

$$\Leftrightarrow 0 < M(\beta)^2 - M(\alpha)^2 < M(\alpha)^{-2} + 2. \tag{6}$$

The existence of such α, β is guaranteed by Lemma 1, since $k(\alpha) = M(\alpha)(4\,D(\alpha) - M(\alpha) - 2)$, by (4). We shall use (5), (6) to determien some families of such examples. Henceforth, we let α, β be such that $M(\alpha) < M(\beta)$ and $D(\alpha) > D(\beta)$. If α, β are each of the form $\alpha^{(k)}$, then each of the constants $k(\alpha), k(\beta)$ is either 0 or 1.

Since $k(\alpha) = k(\beta)$ implies

$$4\,D(\alpha) - 2 = M(\alpha) + k(\alpha)\,M(\alpha)^{-1} = M(\alpha) + k(\beta)\,M(\alpha)^{-1} < M(\beta) + k(\beta)\,M(\beta)^{-1} = 4\,D(\beta) - 2,$$

we must choose $k(\alpha) = 1$ and $k(\beta) = 0$. By Lemma 1, this is equivalent to choosing the largest partial quotient for α to be odd and the largest partial quotient for β to be even.

2 Main Results

Theorem 1. *If α, β are both of the form $[\overline{c_0, c_1}]$, with $c_0 \ge c_1$, then $M(\alpha) < M(\beta)$ and $D(\alpha) > D(\beta)$ if and only if* $(\alpha, \beta) \in \{([\overline{3,1}], [\overline{4,3}]); ([\overline{5,3}], [\overline{4,1}]); ([\overline{5,4}], [\overline{4,1}]); ([\overline{7,3}], [\overline{6,1}])\}$.

Proof. Let $\alpha = [\overline{c_0, c_1}]$, $\beta = [\overline{c_0', c_1'}]$, where $c_0 \ge c_1$, $c_0' \ge c_1'$; we must choose c_0 to be odd and c_0' to be even.

If $|c_0 - c_0'| \ne 1$, then $|c_0 - c_0'| \ge 3$. By the expression for $M([\overline{c_0, c_1}])$ given before Lemma 1, if $c_0' \le c_0 - 3$,

$$M(\beta) - M(\alpha) < (c_0' + 2) - c_0 \le -1,$$

whereas if $c_0' \geq c_0 - 3$,

$$M(\beta) - M(\alpha) > c_0' - (c_0 + 2) \geq -1.$$

This contradicts (5) since $M(\beta) - M(\alpha) < M(\alpha)^{-1} < 1$. Thus, $|c_0 - c_0'| = 1$.

Case 1: ($c_0' = c_0 + 1$) In this case, (6) reduces to

$$0 < 2c_0 + 1 + 4\left(\frac{c_0+1}{c_1'} - \frac{c_0}{c_1}\right) < 2 + \left(c_0^2 + 4\frac{c_0}{c_1}\right)^{-1}. \tag{7}$$

If $c_1 \geq 2$, then

$$2c_0 + 1 + 4\left(\frac{c_0+1}{c_1'} - \frac{c_0}{c_1}\right) \geq 2c_0 + 1 + 4\left(\frac{c_0+1}{c_0+1} - \frac{c_0}{2}\right) = 5 > 2 + \left(c_0^2 + 4\frac{c_0}{c_1}\right)^{-1}.$$

Thus, $c_1 = 1$ and (7) reduces to

$$0 < -2c_0 + 1 + 4\frac{c_0+1}{c_1'} < 2 + (c_0^2 + 4c_0)^{-1}. \tag{8}$$

If $c_1' = 1$ or 2 in (8), there is a contradiction. If $c_1' = 3$, the first inequality in (8) is only satisfied when $1 \leq c_0 \leq 3$. If $c_0 = 1$, then $c_0' = 2$, in contradiction with $c_1' = 3 > 2$. Since c_0 is odd, the only example is obtained when $c_0 = 3$. If $c_1' \geq 4$, w are in conflict with the first inequality of (8), and this case furnishes no further examples. Thus, the only example in this case is $(\alpha, \beta) = ([\overline{3,1}], [\overline{4,3}])$.

Case 2: ($c_0' = c_0 - 1$) In this case, (6) reduces to

$$0 < -2c_0 + 1 + 4\left(\frac{c_0-1}{c_1'} - \frac{c_0}{c_1}\right) < 2 + \left(c_0^2 + 4\frac{c_0}{c_1}\right)^{-1}. \tag{9}$$

If $c_1' \geq 2$, then

$$-2c_0 + 1 + 4\left(\frac{c_0+1}{c_1'} - \frac{c_0}{c_1}\right) \leq -2c_0 + 1 + 4\left(\frac{c_0-1}{2} - \frac{c_0}{c_1}\right) = -1 - 4\frac{c_0}{c_1} < 0.$$

Thus, $c_1' = 1$ and (9) reduces to

$$0 < 2c_0 - 3 - 4\frac{c_0}{c_1} < 2 + (c_0^2 + 4c_0)^{-1}.$$

If $c_1 \leq 2$, then $2c_0 - 3 - 4c_0/c_1 \leq -3$. If $c_1 \geq 5$, then $2c_0 - 3 - 4c_0/c_1 \geq 6c_0/5 - 3 \geq 3$, since $c_0 \geq c_1 \geq 5$. If $c_1 = 3$, then $0 < 2c_0/3 - 3 < 2 + (c_0^2 + 4c_0)^{-1}$, so that $c_0 = 5$ or 7. If $c_1 = 4$, then $0 < c_0 - 3 < 2 + (c_0^2 + 4c_0)^{-1}$, so that $c_0 = 5$. Thus, the three examples in this case are $(\alpha, \beta) = ([\overline{5,3}], [\overline{4,1}])$, $([\overline{5,4}], [\overline{4,1}])$ and $([\overline{7,3}], [\overline{6,1}])$. This completes the proof. □

Table 1 (Theorem 1)

ξ	$M(\xi)$	$D(\xi)$
$[\overline{3,1}] = (3+\sqrt{21})/2$	$\sqrt{21} \approx 4.58258$	$11/2\sqrt{21} + 1/2 \approx 1.70020$
$[\overline{4,3}] = (6+\sqrt{48})/3$	$\sqrt{192}/3 \approx 4.61880$	$16/\sqrt{192} + 1/2 \approx 1.65470$
$[\overline{5,3}] = (15+\sqrt{285})/6$	$\sqrt{285}/3 \approx 5.62731$	$49/2\sqrt{285} + 1/2 \approx 1.95125$
$[\overline{4,1}] = 2+\sqrt{8}$	$\sqrt{32} \approx 5.65685$	$8/\sqrt{32} + 1/2 \approx 1.91421$
$[\overline{5,4}] = (5+\sqrt{30})/2$	$\sqrt{480}/4 \approx 5.47723$	$31/\sqrt{480} + 1/2 \approx 1.91495$
$[\overline{4,1}] = 2+\sqrt{8}$	$\sqrt{32} \approx 5.65685$	$8/\sqrt{32} + 1/2 \approx 1.91421$
$[\overline{7,3}] = (21+\sqrt{525})/6$	$\sqrt{525}/3 \approx 7.63763$	$89/2\sqrt{525} + 1/2 \approx 2.44214$
$[\overline{6,1}] = 3+\sqrt{15}$	$\sqrt{60} \approx 7.74597$	$15/\sqrt{60} + 1/2 \approx 2.43649$

Theorem 2. *If each of α, β is of the form $[\overline{c_0, c_1}]$ or $[\overline{c_0, c_1, c_1}]$, $c_0 \geq c_1$, with α, β of different forms, then $M(\alpha) < M(\beta)$ and $D(\alpha) > D(\beta)$ if and only if $(\alpha, \beta) \in \{([\overline{3,1}], [\overline{4,3,3}])$; $([\overline{5,1}], [\overline{6,2,2}])$; $([\overline{7,1}], [\overline{8,2,2}])$; $([\overline{5,3,3}], [\overline{4,1}])$; $([\overline{11,2,2}], [\overline{10,1}])$; $([\overline{13,2,2}], [\overline{12,1}])\}$.*

The proof of Theorem 2 is similar to that of Theorem 1, and may be found in [5]. Tables similar to Table 1 that verify the results of the various theorems have been omitted.

We observe that $M(\alpha^{(k)}) = c_0 + 2[0, \overline{c_1^{(k)}, c_0}]$ if $c_0 > c_1$. It follows that $\{M(\alpha^{(2k)})\}_{k \geq 0}$ is an increasing sequence, that $\{M(\alpha^{(2k+1)})\}_{k \geq 0}$ is an decreasing sequence, and that $M(\alpha^{(2m+1)}) > M(\alpha^{(2n)})$ for any choice of $m, n \geq 0$. Furthermore,

$$\lim_{n\to\infty} M(\alpha^{(n)}) = c_0 + 2[0, \overline{c_1}] = (c_0 - c_1) + \sqrt{c_1^2 + 4}.$$

Theorem 3. *For any $n \geq 2$, $(\alpha, \beta) \in \{([\overline{3,1}], [\overline{4, 3^{(n)}}])$; $([\overline{5,1}], [\overline{6, 2^{(n)}}])$; $([\overline{7,1}], [\overline{8, 2^{(n)}}])$; $([\overline{5, 3^{(n)}}], [\overline{4,1}])$; $([\overline{11, 2^{(n)}}], [\overline{10,1}])$; $([\overline{13, 2^{(n)}}], [\overline{12,1}])\}$ satisfy the conditions $M(\alpha) < M(\beta)$, $D(\alpha) > D(\beta)$.*

Proof. In view of the observations made above, we need to determine only $M(([\overline{c_0, c_1^{(3)}}])$. The result then follows from the inequalities $M(([\overline{c_0, c_1^{(2)}}]) < M(([\overline{c_0, c_1^{(n)}}]) < M(([\overline{c_0, c_1^{(3)}}])$ for every $n \geq 4$, and from Theorem 2, Lemma 1 and (6). This completes the proof. □

ξ_1	$\xi_2^{(n)}$	$\{M(\xi_1)\}^2$	$\{M(\xi_2^{(2)})\}^2$	$\{M(\xi_2^{(3)})\}^2$	$\{\lim_{n\to\infty} M(\xi_2^{(n)})\}^2$
$[\overline{3,1}]$	$[\overline{4, 3^{(n)}}]$	21.0	21.2	$21.\overline{21}$	21.21110 ...
$[\overline{5,1}]$	$[\overline{6, 2^{(n)}}]$	45.0	46.4	$46.\overline{6}$	46.62741 ...
$[\overline{7,1}]$	$[\overline{8, 2^{(n)}}]$	77.0	77.6	78.0	77.94112 ...
$[\overline{4,1}]$	$[\overline{5, 3^{(n)}}]$	32.0	31.4	$31.\overline{42}$	31.42220 ...
$[\overline{10,1}]$	$[\overline{11, 2^{(n)}}]$	140.0	139.4	140.0	139.91168 ...
$[\overline{12,1}]$	$[\overline{13, 2^{(n)}}]$	192.0	190.6	$191.\overline{3}$	191.22539 ...

Theorem 4. *If each of (α, β) is of the form $[\overline{c_0, c_1, c_1}]$, $c_0 \geq c_1$, then*

$$M(\alpha) < M(\beta) \Longrightarrow D(\alpha) < D(\beta).$$

Proof. If $\alpha = [\overline{c_0, c_1, c_1}]$, $c_0 \geq c_1$ and $\beta = [\overline{c'_0, c'_1, c'_1}]$, $c'_0 \geq c'_1$, then c_0 is odd and c'_0 is even. If $c'_0 \geq c_0 + 1$, then

$$M(\beta)^2 - M(\alpha)^2 = {c'_0}^2 - c_0^2 + 4\left(\frac{c'_0 c'_1 + 1}{{c'_1}^2 + 1} - \frac{c_0 c_1 + 1}{c_1^2 + 1}\right) \geq 2c_0 + 1 + 4\left(1 - \frac{c_0 + 1}{2}\right) = 3,$$

in contradiction to (6).

If $c'_0 \leq c_0 - 1$, then

$$\begin{aligned} M(\beta)^2 - M(\alpha)^2 &= {c'_0}^2 - c_0^2 + 4\left(\frac{c'_0 c'_1 + 1}{{c'_1}^2 + 1} - \frac{c_0 c_1 + 1}{c_1^2 + 1}\right) \\ &\leq -2c_0 + 1 + 4\left(\frac{c'_0 + 1}{2} - 1\right) \\ &\leq -2c_0 + 1 + 4\left(\frac{c_0}{2} - 1\right) = -3, \end{aligned}$$

so that $M(\beta) < M(\alpha)$. This completes the proof. □

V. Drobot [1] observed that $k(\xi) = M(\xi)(4D(\xi) - M(\xi) - 2) = 0$ or 1 depending on whether A is even or odd for $\xi = [1_{m(1)}, A, 1_{m(2)}, A, 1_{m(3)}, A, \ldots]$, where $A > 3$ and $\{m_j\}_{j\geq 1}$ is a non-decreasing sequence of integers tending to infinity. Thus, by the results of Section 1, in order to obtain a counter-example, A must be chosen to be odd if $\alpha = \xi$ and even if $\beta = \xi$.

Since $\{m_j\}_{j\geq 1}$ is a non-decreasing sequence of integers tending to infinity, $M(\xi)$ is computed by taking the limit superior of a sequence with leading partial quotient A. In fact,

$$\begin{aligned} M(\xi) &= \limsup_{i\to\infty} \left([A, 1_{m(i)}, A, 1_{m(i+1)}, \ldots] + [0, 1_{m(i-1)}, A, 1_{m(i-2)}, \ldots, A, 1_{m(1)}]\right) \\ &= A + 2[0, \overline{1}] = A + (\sqrt{5} - 1) \end{aligned} \tag{10}$$

for any $A \geq 1$.

Theorem 5. *If each of (α, β) is of the form $[1_{m(1)}, A, 1_{m(2)}, A, 1_{m(3)}, A, \ldots]$, where $A > 3$ and $\{m_j\}_{j\geq 1}$ is a non-decreasing sequence of integers tending to infinity, then*

$$M(\alpha) < M(\beta) \Longrightarrow D(\alpha) < D(\beta).$$

Proof. By (10), $M(\alpha) \neq M(\beta) \Rightarrow M(\beta) - M(\alpha) \geq 1 > M(\alpha)^{-1}$, which contradicts (5). This completes the proof. □

Theorem 6. *If* $\nu = [\overline{c_0, c_1}]$, *where* $c_0 \geq c_1$, $\xi = [1_{m(1)}, A, 1_{m(2)}, A, 1_{m(3)}, A, \ldots]$, *where* $A > 3$ *and* $\{m_j\}_{j\geq 1}$ *is a non-decreasing sequence of integers tending to infinity, and* (ν, ξ) *satisfy the inequality* $(M(\nu) - M(\xi))(D(\nu) - D(\xi)) < 0$, *then* $(c_0, c_1) \in \{(c_0, 9), 9 \leq c_0 \leq 69, c_0 \text{ odd}; (c_0, 10), 11 \leq c_0 \leq 25, c_0 \text{ odd}; (c_0, 11), 11 \leq c_0 \leq 17, c_0 \text{ odd}; (c_0, 8), 8 \leq c_0 \leq 72, c_0 \text{ even}; (c_0, 7), 8 \leq c_0 \leq 20, c_0 \text{ even}; (c_0, 6), 6 \leq c_0 \leq 10, c_0 \text{ even}; (13, 12); (6, 5)\}$, *and* $A = c_0 - 1$.

The proof of Theorem 6 is similar to that of Theorem 1, and may be found in [5].

Theorem 7. *If* $\nu = [\overline{c_0, c_1, c_1}]$, *where* $c_0 \geq c_1$, $\xi = [1_{m(1)}, A, 1_{m(2)}, A, 1_{m(3)}, A, \ldots]$, *where* $A > 3$ *and* $\{m_j\}_{j\geq 1}$ *is a non-decreasing sequence of integers tending to infinity, and* (ν, ξ) *satisfy the inequality* $(M(\nu) - M(\xi))(D(\nu) - D(\xi)) < 0$, *then* $(c_0, c_1) \in \{(c_0, 9), 9 \leq c_0 \leq 59, c_0 \text{ odd}; (c_0, 10), 11 \leq c_0 \leq 25, c_0 \text{ odd}; (c_0, 11), 11 \leq c_0 \leq 17, c_0 \text{ odd}; (c_0, 8), 8 \leq c_0 \leq 98, c_0 \text{ even}; (c_0, 7), 8 \leq c_0 \leq 22, c_0 \text{ even}; (c_0, 6), 6 \leq c_0 \leq 10, c_0 \text{ even}; (13, 12); (6, 5)\}$, *and* $A = c_0 - 1$.

The proof of Theorem 7 is similar to that of Theorem 1, and is omitted.

We recall that $M(\alpha^{(k)}) = c_0 + 2[0, \overline{c_1^{(k)}, c_0}]$ if $c_0 > c_1$, that $\{M(\alpha^{(2k)})\}_{k\geq 0}$ is an increasing sequence, that $\{M(\alpha^{(2k+1)})\}_{k\geq 0}$ is a decreasing sequence, and that $M(\alpha^{(2m+1)}) > M(\alpha^{(2n)})$ for any choice of $m, n \geq 0$. Furthermore,

$$\lim_{n\to\infty} M(\alpha^{(n)}) = c_0 + 2[0, \overline{c_1}] = (c_0 - c_1) + \sqrt{c_1^2 + 4}.$$

Theorem 8. *If* $\nu = [\overline{c_0, c_1^{(n)}}]$, *where* $c_0 \geq c_1$, $n \geq 1$, *and* $\xi = [1_{m(1)}, A, 1_{m(2)}, A, 1_{m(3)}, A, \ldots]$, *where* $A > 3$ *and* $\{m_j\}_{j\geq 1}$ *is a non-decreasing sequence of integers tending to infinity, and* (ν, ξ) *satisfy the inequality* $(M(\nu) - M(\xi))(D(\nu) - D(\xi)) < 0$, *then* $(c_0, c_1) \in \{(c_0, 9), 9 \leq c_0 \leq 59, c_0 \text{ odd}; (c_0, 10), 11 \leq c_0 \leq 25, c_0 \text{ odd}; (c_0, 11), 11 \leq c_0 \leq 17, c_0 \text{ odd}; (c_0, 8), 8 \leq c_0 \leq 98, c_0 \text{ even}; (c_0, 7), 8 \leq c_0 \leq 22, c_0 \text{ even}; (c_0, 6), 6 \leq c_0 \leq 10, c_0 \text{ even}; (13, 12); (6, 5)\}$, *and* $A = c_0 - 1$.

Proof. In view of the observations made above, the theorem follows from the inequalities $M([\overline{c_0, c_1^{(2)}}] < M([\overline{c_0, c_1^{(n)}}] < M([\overline{c_0, c_1^{(1)}}]$ for every $n \geq 3$, and from Theorems 6 and 7. □

Theorem 9. *For each fixed value of* $n \geq 3$, *there exists* α, β *of the form* $[\overline{c_0, c_1^{(n)}}]$, *where* $c_0 > c_1$, *such that* $(M(\alpha) - M(\beta))(D(\alpha) - D(\beta)) < 0$.

The proof of Theorem 9 is similar to that of Theorem 1, but the computation is more intricate and has been omitted here. It may be noted that we must choose α, β of the form $[\overline{c_0 + 1, c_1^{(n)}}]$, $[\overline{c_0, 1^{(n)}}]$, where $c_0 \geq c_1$ and the largest partial quotient for α is odd.

We observe that if $\alpha = \alpha^{(n)}$, $\beta = \beta^{(n)}$ are chosen as in Theorem 9 and if $(M(\alpha) - M(\beta))(D(\alpha) - D(\beta)) < 0$ for all $n \geq N$, then this inequality must also hold for $\alpha =$

$\lim_{n\to\infty} \alpha^{(n)}$, $\beta = \lim_{n\to\infty} \beta^{(n)}$. This reduces to the choice $\alpha = \lim_{n\to\infty} [\overline{c_0, 1^{(n)}}]$, $\beta = \lim_{n\to\infty} [\overline{c_0+1, c_1^{(n)}}]$, where $c_0 \geq c_1$, c_0 odd. Thus, $M(\alpha) = c_0 - 1 + \sqrt{5}$ and $M(\beta) = c_0 - c_1 + 1 + \sqrt{c_1^2+4}$, so that $c_0 \geq 5$ since $c_0 = 3$ gives $M(\beta)^2 - M(\alpha)^2 = (2+\sqrt{8})^2 - (2+\sqrt{5})^2 > 2 + M(\alpha)^{-2}$. A simple computation yields $5 \leq c_1 \leq 8$, and for each such c_1, the upper limits for c_0 are $5, 9, 21$, and 97, respectively.

If, on the other hand, we choose $\alpha = \lim_{n\to\infty} [\overline{c_0+1, c_1^{(n)}}]$, $\beta = \lim_{n\to\infty} [\overline{c_0, 1^{(n)}}]$, $c_1 \leq c_0$, c_0 even, a simple computation yields $9 \leq c_1 \leq 12$, and for each such c_1, the upper limits for c_0 are $58, 24, 16$, and 12, respectively.

In particular, since $M([\overline{7, 1^{(4)}}])^2 = 67.4 \leq M([\overline{7, 1^{(n)}}])^2 \leq 69 = M([\overline{7, 1^{(3)}}])^2$ for each $n \geq 3$, and $M([\overline{8, 6^{(4)}}])^2 = 69.298220641 \leq M([\overline{8, 6^{(n)}}])^2 \leq 69.298245614 < M([\overline{8, 6^{(3)}}])^2$ for each $n \geq 3$, there exists $\alpha^{(n)}, \beta^{(n)}$ for each $n \geq 3$ such that $(M(\alpha^{(n)}) - M(\beta^{(n)}))(D(\alpha^{(n)}) - D(\beta^{(n)})) < 0$.

We end this paper with a table of values of Markov and dispersion constants of the first sixteen numbers in the Markov spectrum. This list, provided by the referee, contains the first sixteen Markov numbers u_n, the corresponding real irrational number α_n, $\Delta_n := 9u_n^2 - 4$, $M(\alpha_n)$, and $D(\alpha_n)$, and yields another twenty seven counterexamples and suggests the abundance of such.

Table 2. Smallest Markov constants with corresponding dispersion constants

n	u_n	Δ_n	α_n	$M(\alpha_n)$	$D(\alpha_n)$
1	1	5	$[0, \overline{1}]$	2.2360679...	1.1708203...
2	2	32	$[0, \overline{2}]$	2.8284271...	1.2071067...
3	5	221	$[0, \overline{2_2, 1_2}]$	2.9732137...	1.2735737...
4	13	1517	$[0, \overline{2_2, 1_4}]$	2.9960526...	1.2830816...
5	29	7565	$[0, \overline{2_4, 1_2}]$	2.9992071...	1.2760671...
6	34	10400	$[0, \overline{2_2, 1_6}]$	2.9994232...	1.2844645...
7	89	71285	$[0, \overline{2_2, 1_8}]$	2.9999158...	1.2846662...
8	169	257045	$[0, \overline{2_6, 1_2}]$	2.9999766...	1.2761401...
9	194	338720	$[0, \overline{2_2, 1_2, 2_2, 1_4}]$	2.9999822...	1.2835097...
10	233	488597	$[0, \overline{2_2, 1_{10}}]$	2.9999877...	1.2846956...
11	433	1687397	$[0, \overline{2_2, 1_2, 2_4, 1_2}]$	2.9999964...	1.2763673...
12	610	3348896	$[0, \overline{2_2, 1_{12}}]$	2.9999982...	1.2846999...
13	985	8732021	$[0, \overline{2_8, 1_2}]$	2.9999993...	1.2761423...
14	1325	15800621	$[0, \overline{2_2, 1_4, 2_2, 1_6}]$	2.9999996...	1.2845284...
15	1597	22953677	$[0, \overline{2_2, 1_{14}}]$	2.9999997...	1.2847005...
16	2897	75533477	$[0, \overline{2_2, 1_2, 2_2, 1_2, 2_2, 1_4}]$	2.9999999...	1.2835116...

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