A comparison of dispersion and Markoff constants*

by

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0 Introduction

Let \( \{x_n\} \) be a sequence of numbers, \( 0 \leq x_n \leq 1 \). In [3], H. Niederreiter introduced a measure of denseness of such a sequence as follows: For each \( N \geq 1 \), let

\[
d_N = \sup_{0 \leq x_1 \leq 1} \min_{1 \leq n \leq N} |x - x_n|
\]

and define

\[
D(\{x_n\}) = \limsup_{N \to \infty} N d_N.
\]

In particular, for irrational \( \alpha \), the dispersion constant \( D(\alpha) \) is defined by \( D(\{n\alpha \mod 1\}) \). It is well known that, for irrational \( \alpha \), the Markoff constant \( M(\alpha) \) is defined by

\[
M(\alpha)^{-1} = \liminf_{n \to \infty} n ||n\alpha||,
\]

where \( ||x|| \) denotes the distance from \( x \) to the nearest integer.

In [3], Niederreiter asks if \( M(\alpha) < M(\beta) \) implies \( D(\alpha) < D(\beta) \). V. Drobot [1] has shown this to be false by producing a counterexample of two quadratic irrationals, both with continued fraction expansions with period length nine. In this paper, we classify some infinite families of pairs \( (\alpha, \beta) \) of irrational numbers that satisfy \( M(\alpha) < M(\beta) \) and \( D(\alpha) > D(\beta) \).

We first outline the method of V. Drobot [1] to compute \( D(\alpha) \) for quadratic irrationals \( \alpha \). If \( \alpha \) is a real irrational number with continued fraction expansion \( \alpha = [a_0; a_1, a_2, \ldots] \), let

\[
\lambda_i = [0; a_i, a_{i-1}, \ldots, a_1], \quad \Lambda_i = [a_{i+1}; a_{i+2}, \ldots], \quad M_i = \lambda_i + \Lambda_i.
\]

We define

\[
\psi_i(x) = \frac{1}{M_i}[-x^2 + (\Lambda_i - \lambda_i - 1)x + \Lambda_i(1 + \lambda_i)],
\]

\[
x_i = \frac{\Lambda_i - \lambda_i - 1}{2}, \quad n_i \text{ is the integer closest to } x_i.
\]

*Appeared in Acta Arithmetica, 63.3 (1993), 193–203
Then Drobot [1] has shown that \( D(\alpha) = \limsup_{i \to \infty} \psi_i(n_i) \). In particular, if \( \alpha \) has a periodic continued fraction expansion, there are only finitely many choices for \( i \) and taking the lim sup reduces to taking the maximum of the values \( \psi_i(n_i) = D_i \) of the quadratic polynomial \( \psi_i(x) \).

In view of [3], Theorem 6, p. 1197, it is no restriction to suppose that this expansion is purely periodic and that \( a_0 \geq 1 \):

\[
\alpha = \left[ a_0; a_1; \ldots; a_{k-1} \right], a_0 \geq 1.
\]

We extend the periodic sequence

\[
c_0; c_1, c_2, \ldots = a_0; a_1, a_2, \ldots, a_{k-1}, a_0, a_1, \ldots
\]

periodically in the other direction as well, that is,

\[
\ldots c_{-2}, c_{-1}, c_0; c_1, c_2, \ldots = \ldots a_0, a_1, \ldots, a_{k-1}, a_0; a_1, \ldots, a_k, a_0, a_1, \ldots
\]

Then, with \( \overline{\lambda}_i = [0; c_i, c_{i-1}, \ldots], \overline{\lambda}_i = [c_{i+1}; c_{i+2}, \ldots], \overline{M}_i = \overline{\lambda}_i + \overline{\lambda}_i, \) and similar definitions for \( \overline{\psi}_i(x), \overline{\pi}_i, \overline{\pi}_i \), one has (see [1], p. 93) \( D(\alpha) = \max_{0 \leq i \leq k-1} \overline{\psi}_i(n_i) \). In this paper, we only deal with the quadratic irrational case, therefore we may and do, omit the bar in the above notation without causing any confusion.

We use this method to determine \( D(\alpha) \) for \( \alpha \) that has a purely periodic continued fraction expansion. Let

\[
\alpha = \frac{A + \sqrt{D}}{B},
\]

where

\[
A = p_{k-1}(\alpha) - q_{k-2}(\alpha), \quad B = 2q_{k-1}(\alpha), \quad D = (p_{k-1}(\alpha) + q_{k-2}(\alpha))^2 - 4(-1)^k;
\]

here \( p_n(\alpha)/q_n(\alpha) \) is the \( n \)th convergent to \([a_0; a_1, \ldots, a_{k-1}]\). If we set \( \lambda_i = (a + \sqrt{D})/b \), so that \( M_i = 2\sqrt{D}/b \) and \( 2x_i = (2a - b)/b \).

In particular, if \( n_i = 0 \), then

\[
M_i D_i = \Lambda_i + \Lambda_i \lambda_i = \frac{a + \sqrt{D}}{b} + \frac{D - a^2}{b^2},
\]

so that

\[
(A) \quad D_i = \frac{1}{M_i} \left( \frac{a}{b} + \frac{D - a^2}{b^2} \right) + \frac{1}{2}
\]

\[
= \frac{q_{k-1}(\Lambda_i)(p_{k-1}(\Lambda_i) + q_{k-2}(\Lambda_i))^2 - 4(-1)^k}{2q_{k-1}(\Lambda_i)} + \frac{1}{2}
\]

\[
\times \left( \frac{p_{k-1}(\Lambda_i) - q_{k-2}(\Lambda_i)}{q_{k-1}(\Lambda_i)} \right) + \frac{1}{2}
\]

\[
= \left( \frac{p_{k-1}(\Lambda_i) - q_{k-2}(\Lambda_i)}{2q_{k-1}(\Lambda_i)} \right) + \frac{1}{2}
\]

If \( n_i = 1 \), then

\[
M_i D_i = 2x_i - 1 + \Lambda_i + \Lambda_i \lambda_i = \frac{2a - b}{b} + \frac{a + \sqrt{D}}{b} + \frac{D - a^2}{b^2},
\]
so that

\[(B) \quad D_i = \left[ (p_{k-1}(\Lambda_i) + q_{k-2}(\Lambda_i))^2 - 4(-1)^k \right]^{-1/2}
\]

\times \left( \frac{p_{k-1}(\Lambda_i) - q_{k-2}(\Lambda_i)}{2} + p_{k-2}(\Lambda_i) + (p_{k-1}(\Lambda_i) - q_{k-2}(\Lambda_i) - 2q_{k-1}(\Lambda_i)) + \frac{1}{2} \right)

\[= \left[ (p_{k-1}(\Lambda_i) + q_{k-2}(\Lambda_i))^2 - 4(-1)^k \right]^{-1/2}
\]

\times \left( \frac{3}{2}(p_{k-1}(\Lambda_i) - q_{k-2}(\Lambda_i)) + p_{k-2}(\Lambda_i) - 2q_{k-1}(\Lambda_i) \right) + \frac{1}{2}

Thus, if \(n_i\) only takes the values 0 and 1 for \(i = 0, \ldots, k - 1\), then \(D(\alpha)\) is computed by taking the maximum of the relevant expressions given by equations (A), (B).

For each \(k \geq 1\), let \(\alpha^{(k)} = [c_0; c_1, c_1, \ldots, c_1]\), with \(k\) occurrences of \(c_1\) and where \(c_0 > c_1\). For each \(k \geq 1\), we set

\[\alpha^{(k)} = \Lambda^{(k)} = [c_0; c_1, c_1, \ldots, c_1], \quad \lambda^{(k)} = [0; c_1, c_1, \ldots, c_1, c_0]\]

with \(k\) occurrences of \(c_1\).

If \(\Lambda^{(k)} = (a_k + \sqrt{d_k})/b_k\), then \(\lambda^{(k)} = (-a_k + \sqrt{d_k})/b_k\), so that

\[M^{(k)} = \Lambda^{(k)} + \lambda^{(k)} = 2\frac{\sqrt{d_k}}{b_k}.
\]

In particular, for a fixed \(k\), the numerators and denominators \(p^{(k)}_\ell\) and \(q^{(k)}_\ell\) of \(\alpha^{(k)}\) satisfy

\[p^{(k)}_{-1} = 1, \quad q^{(k)}_{-1} = 0, \quad p^{(k)}_0 = c_0, \quad q^{(k)}_0 = 1,
\]

\[p^{(k)}_\ell = c_1 p^{(k)}_{\ell-1} + p^{(k)}_{\ell-2}, \quad q^{(k)}_\ell = c_1 q^{(k)}_{\ell-1} + q^{(k)}_{\ell-2}, \quad 1 \leq \ell \leq k.
\]

Since these are second order linear recurrence equations with constant coefficients, it follows from well known facts (see [4], pp. 121-122, for instance) that

\[(r_2 - r_1)p^{(k)}_\ell = c_0 (r^{\ell+1}_2 - r^{\ell+1}_1) + r^\ell_2 - r^\ell_1, \quad (r_2 - r_1)q^{(k)}_\ell = r^{\ell+1}_2 - r^{\ell+1}_1, \quad \ell = 0, 1, \ldots, k,
\]

where \(r_1, r_2\) satisfy the equation \(r^2 - c_1 r - 1 = 0\).

Since \(a_k = p^{(k)}_k - q^{(k)}_{k-1}\), \(b_k = 2q^{(k)}_k\), and \(d_k = (p^{(k)}_k + q^{(k)}_{k-1})^2 + 4(-1)^k\), it follows that

\[M_k(\alpha^{(k)}) = \sqrt{\frac{(p^{(k)}_k + q^{(k)}_{k-1})^2 + 4(-1)^k}{q^{(k)}_k}},
\]

so that

\[M_k(\alpha^{(k)})^2 = c^2_0 + 4(r^{k+1}_2 - r^{k+1}_1)^2 \left\{ c_0 (r^{\ell+1}_2 - r^{\ell+1}_1)(r^{\ell+1}_2 - r^{\ell+1}_1) + (r^\ell_2 - r^\ell_1)^2
\]

\[+(-1)^k(r_2 - r_1) \right\}

\[= c^2_0 + 4(r^{k+1}_2 - r^{k+1}_1)^2 \left\{ c_0 (r^{2k+1}_2 + r^{2k+1}_1 + (-1)^k c_1) + (r^k_2 - r^k_1)^2
\]

\[+(-1)^k(c^2_0 + 4) \right\}.
\]
With the previous notation, we may now write $\alpha^{(k)} = \Lambda^{(k)} = \Lambda^k(\alpha^{(k)})$ and $\lambda^{(k)} = \lambda^k(\alpha^{(k)})$. For an $\alpha$ of the special form $\alpha^{(k)} = [c_0; c_1, \ldots, c_1]$, with $k$ occurrences of $c_1$, we have $M(\alpha^{(k)}) = \max_{0 \leq i \leq k-1} M_i$ (see [2], formula (11), p. 29). Since $c_0 > c_1$, we have

$$M_k(\alpha^{(k)}) = c_0 + 2[0; c_1, \ldots, c_1, c_0] > c_1 + 2[0; c_1, c_0, c_1, \ldots, c_1] = M_1(\alpha^{(k)}) \geq M_k(\alpha^{(k)}) \text{ for } i \neq k.$$  

Thus,

$$M(\alpha^{(k)}) = M_k(\alpha^{(k)}).$$  

Using (3), it is easy to see that

$$M([c_0]) = \sqrt{c_0^2 + 4}, \quad M([c_0, c_1]) = \sqrt{c_0^2 + 4\frac{c_0}{c_1}}, \quad \text{and} \quad M([c_0, c_1, c_1]) = \sqrt{c_0^2 + 4\frac{c_0c_1 + 1}{c_1^2 + 1}}.$$  

We observe that for each $k \geq 1$, $\Lambda^{(k)} - \lambda^{(k)} = c_0$, so that $n^{(k)} = [c_0/2]$, where $n^{(k)}$ is defined to be the integer closest to $(\Lambda^{(k)} - \lambda^{(k)} - 1)/2$.

**Lemma 1.** For each $k \geq 1$,

$$M(\alpha^{(k)}) \{4 D(\alpha^{(k)}) - M(\alpha^{(k)}) - 2\} = \begin{cases} 0 & \text{if } c_0 \text{ is even;} \\ 1 & \text{if } c_0 \text{ is odd.} \end{cases}$$

**Proof.** We recall that for $\alpha$ with a periodic continued fraction expansion,

$$D(\alpha) = \max_i \psi_i(n_i) = \max_i \frac{1}{\Lambda_i} \left\{ -n_i^2 + (\Lambda_i - \lambda_i - 1)n_i + \Lambda_i(1 + \lambda_i) \right\}.$$  

Since $n_i$ is the integer closest to $(\Lambda_i - \lambda_i - 1)/2$, the expression for $\psi_i(n_i)$ is an increasing function of $\Lambda_i$, so that $D(\alpha) = \psi_i(n_k)$, where $n_k = n^{(k)}$. Also, since $M_i = \lambda_i + \Lambda_i$, we have

$$M(\alpha^{(k)}) D(\alpha^{(k)}) = -\left(n^{(k)}\right)^2 + (\Lambda^{(k)} - \lambda^{(k)} - 1)n^{(k)} + \Lambda^{(k)}(\lambda^{(k)} + 1)$$

$$= -\left[c_0^2/2\right] + (c_0 - 1)\left[c_0/2\right] + \frac{d_k - a_k^2}{b_k^2} + \frac{a_k + \sqrt{d_k}}{b_k}$$

$$= -\left[c_0^2/2\right] + (c_0 - 1)\left[c_0/2\right] + \frac{a_k - a_k^2}{b_k} - \frac{a_k^2}{b_k^2} + \frac{1}{2M(\alpha^{(k)})} + \frac{1}{4M(\alpha^{(k)})^2},$$  

so that

$$M(\alpha^{(k)}) \{4 D(\alpha^{(k)}) - M(\alpha^{(k)}) - 2\} = 4 \left\{ \frac{a_k}{b_k} - \left(\frac{c_0}{2}\right)^2 + (c_0 - 1)\left[\frac{c_0}{2}\right] - \left[\frac{c_0}{2}\right]^2 \right\}$$

$$= 4 \left\{ \frac{c_0}{2} - \left(\frac{c_0}{2}\right)^2 + (c_0 - 1)\left[\frac{c_0}{2}\right] - \left[\frac{c_0}{2}\right]^2 \right\}$$

$$= \begin{cases} 0 & \text{if } c_0 \text{ is even;} \\ 1 & \text{if } c_0 \text{ is odd.} \end{cases}$$

\[
\square
\]
1  **A condition for** $M(\alpha) < M(\beta)$ **and** $D(\alpha) > D(\beta)$

We recall a result of Drobot [1]: $M(\alpha) \leq 4D(\alpha) - 2 \leq M(\alpha) + M(\alpha)^{-1}$ for any real $\alpha$. Thus, with each real $\alpha$, we may associate a constant $k(\alpha) = k \in [0,1]$ such that $4D(\alpha) - 2 = M(\alpha) + kM(\alpha)^{-1}$.

Thus, we may define

\begin{equation}
(4) \quad k(\alpha) = M(\alpha)(4D(\alpha) - M(\alpha) - 2).
\end{equation}

Suppose that $M(\alpha) < M(\beta)$ but $D(\alpha) > D(\beta)$. Now,

\begin{equation}
(5) \quad M(\alpha) < M(\beta) \iff M(\alpha) + k(\alpha)M(\alpha)^{-1} > M(\beta) + k(\beta)M(\beta)^{-1},
\end{equation}

for some $k(\alpha), k(\beta) \in [0,1]$. If we can find $\alpha$ such that $k(\alpha) = 1$ and $\beta$ such that $k(\beta) = 0$, then this reduces to

\begin{equation}
(6) \quad 0 < M(\beta)^2 - M(\alpha)^2 < M(\alpha)^{-2} + 2.
\end{equation}

The existence of such $\alpha, \beta$ is guaranteed by Lemma 1, since $k(\alpha) = M(\alpha)(4D(\alpha) - M(\alpha) - 2)$, by (4). We shall use (5), (6) to determine some families of such examples. Henceforth, we let $\alpha, \beta$ be such that $M(\alpha) < M(\beta)$ and $D(\alpha) > D(\beta)$. If $\alpha, \beta$ are each of the form $\alpha^{(k)}$, then each of the constants $k(\alpha), k(\beta)$ is either 0 or 1.

Since $k(\alpha) = k(\beta)$ implies

\begin{equation}
4D(\alpha) - 2 = M(\alpha) + k(\alpha)M(\alpha)^{-1} = M(\alpha) + k(\beta)M(\alpha)^{-1} < M(\beta) + k(\beta)M(\beta)^{-1} = 4D(\beta) - 2,
\end{equation}

we must choose $k(\alpha) = 1$ and $k(\beta) = 0$. By Lemma 1, this is equivalent to choosing the largest partial quotient for $\alpha$ to be odd and the largest partial quotient for $\beta$ to be even.

2  **Main Results**

**Theorem 1.** If $\alpha, \beta$ are both of the form $\frac{[c_0, c_1]}{c_0}$, with $c_0 \geq c_1$, then $M(\alpha) < M(\beta)$ and $D(\alpha) > D(\beta)$ if and only if $(\alpha, \beta) \in \{([3, 1], [4, 3]); ([5, 3], [4, 1]); ([5, 4], [4, 1]); ([7, 3], [6, 1])\}$.

**Proof.** Let $\alpha = \frac{[c_0, c_1]}{c_0}, \beta = \frac{[c'_0, c'_1]}{c'_0}$, where $c_0 \geq c_1, c'_0 \geq c'_1$; we must choose $c_0$ to be odd and $c'_0$ to be even.

If $|c_0 - c'_0| \neq 1$, then $|c_0 - c'_0| \geq 3$. By the expression for $M([c_0, c_1])$ given before Lemma 1, if $c'_0 \leq c_0 - 3$,

\[ M(\beta) - M(\alpha) < (c'_0 + 2) - c_0 \leq -1, \]
Thus, $c_0 = c_0 + 1$ then $0 < -2c_0 + 1 + 4\left(\frac{c_0 + 1}{c_1} - \frac{c_0}{c_1}\right) < 2 + \left(\frac{c_0^2 + 4c_0}{c_1}\right)^{-1}$.

If $c_1 \geq 2$, then

$$-2c_0 + 1 + 4\left(\frac{c_0 + 1}{c_1} - \frac{c_0}{c_1}\right) \leq -2c_0 + 1 + 4\left(\frac{c_0 - 1}{2} - \frac{c_0}{c_1}\right) = -1 - 4\frac{c_0}{c_1} < 0.$$  

Thus, $c_1 = 1$ and (9) reduces to

$$0 < 2c_0 - 3 - 4\frac{c_0}{c_1} < 2 + \left(\frac{c_0^2 + 4c_0}{c_1}\right)^{-1}.$$  

If $c_1 \leq 2$, then $2c_0 - 3 - 4c_0/c_1 \leq -3$. If $c_1 \geq 5$, then $2c_0 - 3 - 4c_0/c_1 \geq 6c_0/5 - 3 \geq 3$, since $c_0 \geq c_1 \geq 5$. If $c_1 = 3$, then $0 < 2c_0/3 - 3 < 2 + (c_0^2 + 4c_0)^{-1}$, so that $c_0 = 5$ or 7. If $c_1 = 4$, then $0 < c_0 - 3 < 2 + (c_0^2 + 4c_0)^{-1}$, so that $c_0 = 5$. Thus, the three examples in this case are $(\alpha, \beta) = ([3, 1], [4, 3])$, $([5, 4], [4, 1])$ and $(7, 3), [6, 1])$. This completes the proof. \qed
Table 1 (Theorem 1)

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$M(\xi)$</th>
<th>$D(\xi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[3,1] = (3 + \sqrt{21})/2$</td>
<td>$\sqrt{21} \approx 4.58258$</td>
<td>$11/2 \sqrt{21} + 1/2 \approx 1.70020$</td>
</tr>
<tr>
<td>$[4,3] = (6 + \sqrt{48})/3$</td>
<td>$\sqrt{192}/3 \approx 4.61880$</td>
<td>$16/\sqrt{192} + 1/2 \approx 1.65470$</td>
</tr>
<tr>
<td>$[5,3] = (15 + \sqrt{285})/6$</td>
<td>$\sqrt{285}/3 \approx 5.62731$</td>
<td>$49/2 \sqrt{285} + 1/2 \approx 1.95125$</td>
</tr>
<tr>
<td>$[4,1] = 2 + \sqrt{8}$</td>
<td>$\sqrt{32} \approx 5.65685$</td>
<td>$8/\sqrt{32} + 1/2 \approx 1.91421$</td>
</tr>
<tr>
<td>$[5,4] = (5 + \sqrt{30})/2$</td>
<td>$\sqrt{480}/4 \approx 5.47723$</td>
<td>$31/\sqrt{480} + 1/2 \approx 1.91495$</td>
</tr>
<tr>
<td>$[4,1] = 2 + \sqrt{8}$</td>
<td>$\sqrt{32} \approx 5.65685$</td>
<td>$8/\sqrt{32} + 1/2 \approx 1.91421$</td>
</tr>
<tr>
<td>$[7,3] = (21 + \sqrt{525})/6$</td>
<td>$\sqrt{525}/3 \approx 7.63763$</td>
<td>$89/2 \sqrt{525} + 1/2 \approx 2.44214$</td>
</tr>
<tr>
<td>$[6,1] = 3 + \sqrt{15}$</td>
<td>$\sqrt{60} \approx 7.74597$</td>
<td>$15/\sqrt{60} + 1/2 \approx 2.43649$</td>
</tr>
</tbody>
</table>

Theorem 2. If each of $\alpha, \beta$ is of the form $[c_0, c_1]$ or $[c_0, c_1]$, $c_0 \geq c_1$, with $\alpha, \beta$ of different forms, then $M(\alpha) < M(\beta)$ and $D(\alpha) > D(\beta)$ if and only if $(\alpha, \beta) \in \{(3, 1), (4, 3), (5, 3), (4, 1), (5, 4), (7, 3), (6, 1)\}$. Tables similar to Table 1 that verify the results of the various theorems have been omitted.

We observe that $M(\alpha(k)) = c_0 + 2[0, c_1(0), c_0]$ if $c_0 > c_1$. It follows that $\{M(\alpha(2k))\}_{k \geq 0}$ is an increasing sequence, that $\{M(\alpha(2k+1))\}_{k \geq 0}$ is a decreasing sequence, and that $M(\alpha(2m+1)) > M(\alpha(2n))$ for any choice of $m, n \geq 0$. Furthermore,

$$\lim_{n \to \infty} M(\alpha(n)) = c_0 + 2[0, \tau_1] = (c_0 - c_1) + \sqrt{c_1^2 + 4}.$$ 

Theorem 3. For any $n \geq 2$, $(\alpha, \beta) \in \{(3, 1), (4, 3(n)), (5, 1), (6, 2(n)), (7, 1), (8, 2(n)), (9, 1), (10, 1), (11, 2(n)), (12, 1)\}$ satisfy the conditions $M(\alpha) < M(\beta)$, $D(\alpha) > D(\beta)$.

Proof. In view of the observations made above, we need to determine only $M((c_0, c_1^{(n)}))$. The result then follows from the inequalities $M((c_0, c_1^{(n)})) < M((c_0, c_1^{(n)})) < M((c_0, c_1^{(n)}))$ for every $n \geq 4$, and from Theorem 2, Lemma 1 and (6). This completes the proof.

<table>
<thead>
<tr>
<th>$\xi_1$</th>
<th>$\xi_2^{(n)}$</th>
<th>${M(\xi_1)}^2$</th>
<th>${M(\xi_2^{(2)})}^2$</th>
<th>${M(\xi_3^{(3)})}^2$</th>
<th>$\lim_{n \to \infty} M(\xi_2^{(n)})^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[3,1]$</td>
<td>$4, 3(n)$</td>
<td>21.0</td>
<td>21.2</td>
<td>21.21</td>
<td>21.21110...</td>
</tr>
<tr>
<td>$[5,1]$</td>
<td>$6, 2(n)$</td>
<td>45.0</td>
<td>46.4</td>
<td>46.6</td>
<td>46.62741...</td>
</tr>
<tr>
<td>$[7,1]$</td>
<td>$8, 2(n)$</td>
<td>77.0</td>
<td>77.6</td>
<td>78.0</td>
<td>77.94112...</td>
</tr>
<tr>
<td>$[4,1]$</td>
<td>$5, 3(n)$</td>
<td>32.0</td>
<td>31.4</td>
<td>31.32</td>
<td>31.42220...</td>
</tr>
<tr>
<td>$[10,1]$</td>
<td>$11, 2(n)$</td>
<td>140.0</td>
<td>139.4</td>
<td>140.0</td>
<td>139.91168...</td>
</tr>
<tr>
<td>$[12,1]$</td>
<td>$13, 2(n)$</td>
<td>192.0</td>
<td>190.6</td>
<td>191.33</td>
<td>191.22539...</td>
</tr>
</tbody>
</table>
Theorem 4. If each of \((\alpha, \beta)\) is of the form \([c_0, c_1, c_1], \ c_0 \geq c_1\), then
\[
M(\alpha) < M(\beta) \implies D(\alpha) < D(\beta).
\]

Proof. If \(\alpha = [c_0, c_1, c_1], \ c_0 \geq c_1\) and \(\beta = [c_0', c_1', c_1'], \ c_0' \geq c_1'\), then \(c_0\) is odd and \(c_0'\) is even.
If \(c_0' \geq c_0 + 1\), then
\[
M(\beta)^2 - M(\alpha)^2 = c_0'^2 - c_0^2 + 4\left(\frac{c_0'c_1' + 1}{c_1'^2 + 1} - \frac{c_0c_1 + 1}{c_1^2 + 1}\right) \geq 2c_0 + 1 + 4\left(1 - \frac{c_0 + 1}{2}\right) = 3,
\]
in contradiction to (6).
If \(c_0' \leq c_0 - 1\), then
\[
M(\beta)^2 - M(\alpha)^2 = \leq -2c_0 + 1 + 4\left(\frac{c_0 + 1}{2} - 1\right) = -3,
\]
so that \(M(\beta) < M(\alpha)\). This completes the proof. \(\square\)

V. Drobot [1] observed that \(k(\xi) = M(\xi)(4D(\xi) - M(\xi) - 2) = 0\) or \(1\) depending on whether \(A\) is even or odd for \(\xi = [1_{m(1)}, A, 1_{m(2)}, A, 1_{m(3)}, A, \ldots]\), where \(A > 3\) and \(\{m_j\}_{j \geq 1}\) is a non-decreasing sequence of integers tending to infinity. Thus, by the results of Section 1, in order to obtain a counter-example, \(A\) must be chosen to be odd if \(\alpha = \xi\) and even if \(\beta = \xi\).

Since \(\{m_j\}_{j \geq 1}\) is a non-decreasing sequence of integers tending to infinity, \(M(\xi)\) is computed by taking the limit superior of a sequence with leading partial quotient \(A\). In fact,
\[
M(\xi) = \limsup_{i \to \infty} (\lfloor A, 1_{m(i)}, A, 1_{m(i+1)}, \ldots \rfloor + \lfloor 0, 1_{m(i-1)}, A, 1_{m(i-2)}, \ldots, A, 1_{m(1)} \rfloor) = A + 2[0, 1] = A + (\sqrt{5} - 1)
\]
for any \(A \geq 1\).

Theorem 5. If each of \((\alpha, \beta)\) is of the form \([1_{m(1)}, A, 1_{m(2)}, A, 1_{m(3)}, A, \ldots]\), where \(A > 3\) and \(\{m_j\}_{j \geq 1}\) is a non-decreasing sequence of integers tending to infinity, then
\[
M(\alpha) < M(\beta) \implies D(\alpha) < D(\beta).
\]

Proof. By (10), \(M(\alpha) \neq M(\beta) \Rightarrow M(\beta) - M(\alpha) \geq 1 > M(\alpha)^{-1}\), which contradicts (5). This completes the proof. \(\square\)
Theorem 6. If \( \nu = [c_0, c_1] \), where \( c_0 \geq c_1 \), \( \xi = [1_{m(1)}, A, 1_{m(2)}, A, 1_{m(3)}, A, \ldots] \), where \( A > 3 \) and \( \{m_j\}_{j \geq 1} \) is a non-decreasing sequence of integers tending to infinity, and \( (\nu, \xi) \) satisfy the inequality \( (M(\nu) - M(\xi))(D(\nu) - D(\xi)) < 0 \), then \( (c_0, c_1) \in \{ (c_0, 9), 9 \leq c_0 \leq 69, c_0 \text{ odd}; (c_0, 10), 11 \leq c_0 \leq 25, c_0 \text{ odd}; (c_0, 11), 11 \leq c_0 \leq 17, c_0 \text{ odd}; (c_0, 8), 8 \leq c_0 \leq 72, c_0 \text{ even}; (c_0, 7), 8 \leq c_0 \leq 20, c_0 \text{ even}; (c_0, 6), 6 \leq c_0 \leq 10, c_0 \text{ even}; (13, 12); (6, 5) \} \), and \( A = c_0 - 1 \).

The proof of Theorem 6 is similar to that of Theorem 1, and may be found in [5].

Theorem 7. If \( \nu = [c_0, c_1, c_2] \), where \( c_0 \geq c_1 \), \( \xi = [1_{m(1)}, A, 1_{m(2)}, A, 1_{m(3)}, A, \ldots] \), where \( A > 3 \) and \( \{m_j\}_{j \geq 1} \) is a non-decreasing sequence of integers tending to infinity, and \( (\nu, \xi) \) satisfy the inequality \( (M(\nu) - M(\xi))(D(\nu) - D(\xi)) < 0 \), then \( (c_0, c_1) \in \{ (c_0, 9), 9 \leq c_0 \leq 59, c_0 \text{ odd}; (c_0, 10), 11 \leq c_0 \leq 25, c_0 \text{ odd}; (c_0, 11), 11 \leq c_0 \leq 17, c_0 \text{ odd}; (c_0, 8), 8 \leq c_0 \leq 98, c_0 \text{ even}; (c_0, 7), 8 \leq c_0 \leq 22, c_0 \text{ even}; (c_0, 6), 6 \leq c_0 \leq 10, c_0 \text{ even}; (13, 12); (6, 5) \} \), and \( A = c_0 - 1 \).

The proof of Theorem 7 is similar to that of Theorem 1, and is omitted.

We recall that \( M(\alpha^{(k)}) = c_0 + 2[0, c_1^{(k)}] \) if \( c_0 > c_1 \), that \( \{M(\alpha^{(2k+1)})\}_{k \geq 0} \) is an increasing sequence, that \( \{M(\alpha^{(2k+1)})\}_{k \geq 0} \) is a decreasing sequence, and that \( M(\alpha^{(2m+1)}) > M(\alpha^{(2m)}) \) for any choice of \( m, n \geq 0 \). Furthermore,

\[
\lim_{n \to \infty} M(\alpha^{(n)}) = c_0 + 2[0, \tau_1] = (c_0 - c_1) + \sqrt{c_1^2 + 4}.
\]

Theorem 8. If \( \nu = [c_0, c_1^{(n)}] \), where \( c_0 \geq c_1 \), \( n \geq 1 \), and \( \xi = [1_{m(1)}, A, 1_{m(2)}, A, 1_{m(3)}, A, \ldots] \), where \( A > 3 \) and \( \{m_j\}_{j \geq 1} \) is a non-decreasing sequence of integers tending to infinity, and \( (\nu, \xi) \) satisfy the inequality \( (M(\nu) - M(\xi))(D(\nu) - D(\xi)) < 0 \), then \( (c_0, c_1) \in \{ (c_0, 9), 9 \leq c_0 \leq 59, c_0 \text{ odd}; (c_0, 10), 11 \leq c_0 \leq 25, c_0 \text{ odd}; (c_0, 11), 11 \leq c_0 \leq 17, c_0 \text{ odd}; (c_0, 8), 8 \leq c_0 \leq 98, c_0 \text{ even}; (c_0, 7), 8 \leq c_0 \leq 22, c_0 \text{ even}; (c_0, 6), 6 \leq c_0 \leq 10, c_0 \text{ even}; (13, 12); (6, 5) \} \), and \( A = c_0 - 1 \).

Proof. In view of the observations made above, the theorem follows from the inequalities

\[
M([c_0, c_1^{(2)}]) < M([c_0, c_1^{(n)}]) < M([c_0, c_1^{(1)}])
\]

for every \( n \geq 3 \), and from Theorems 6 and 7. \( \square \)

Theorem 9. For each fixed value of \( n \geq 3 \), there exists \( \alpha, \beta \) of the form \([c_0, c_1^{(n)}]\), where \( c_0 > c_1 \), such that \( (M(\alpha) - M(\beta))(D(\alpha) - D(\beta)) < 0 \).

The proof of Theorem 9 is similar to that of Theorem 1, but the computation is more intricate and has been omitted here. It may be noted that we must choose \( \alpha, \beta \) of the form

\[
[c_0 + 1, c_1^{(n)}], [c_0, 1^{(n)}]
\]

where \( c_0 \geq c_1 \) and the largest partial quotient for \( \alpha \) is odd.

We observe that if \( \alpha = \alpha^{(n)}, \beta = \beta^{(n)} \) are chosen as in Theorem 9 and if \( (M(\alpha) - M(\beta))(D(\alpha) - D(\beta)) < 0 \) for all \( n \geq N \), then this inequality must also hold for \( \alpha = \)
\[ \lim_{n \to \infty} \alpha^{(n)}, \beta = \lim_{n \to \infty} \beta^{(n)}. \] This reduces to the choice \( \alpha = \lim_{n \to \infty} [c_0, 1^{(n)}], \beta = \lim_{n \to \infty} [c_0 + 1, c_1^{(n)}] \), where \( c_0 \geq c_1, c_0 \) odd. Thus, \( M(\alpha) = c_0 - 1 + \sqrt{5} \) and \( M(\beta) = c_0 - c_1 - 1 + \sqrt{c_1^2 + 4} \), so that \( c_0 \geq 5 \) since \( c_0 = 3 \) gives \( M(\beta)^2 - M(\alpha)^2 = (2 + \sqrt{8})^2 - (2 + \sqrt{5})^2 > 2 + M(\alpha)^{-2} \). A simple computation yields \( 5 \leq c_1 \leq 8 \), and for each such \( c_1 \), the upper limits for \( c_0 \) are 5, 9, 21, and 97, respectively.

If, on the other hand, we choose \( \alpha = \lim_{n \to \infty} [c_0 + 1, c_1^{(n)}], \beta = \lim_{n \to \infty} [c_0, 1^{(n)}], c_1 \leq c_0, c_0 \) even, a simple computation yields \( 9 \leq c_1 \leq 12 \), and for each such \( c_1 \), the upper limits for \( c_0 \) are 58, 24, 16, and 12, respectively.

In particular, since \( M([7, 1^{(4)})]^2 = 67.4 \leq M([7, 1^{(2)})]^2 \leq 69 = M([7, 1^{(3)})]^2 \) for each \( n \geq 3 \), and \( M([8, 6^{(4)})]^2 = 69.298220641 \leq M([8, 6^{(2)})]^2 \leq 69.298245614 < M([8, 6^{(3)})]^2 \) for each \( n \geq 3 \), there exists \( \alpha^{(n)}, \beta^{(n)} \) for each \( n \geq 3 \) such that \( (M(\alpha^{(n)}) - M(\beta^{(n)}))D(\alpha^{(n)}) - D(\beta^{(n)}) < 0 \).

We end this paper with a table of values of Markov and dispersion constants of the first sixteen numbers in the Markov spectrum. This list, provided by the referee, contains the first sixteen Markov numbers \( u_n \), the corresponding real irrational number \( \alpha_n, \Delta_n := 9u_n^2 - 4, M(\alpha_n), \) and \( D(\alpha_n), \) and yields another twenty seven counterexamples and suggests the abundance of such.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( u_n )</th>
<th>( \Delta_n )</th>
<th>( \alpha_n )</th>
<th>( M(\alpha_n) )</th>
<th>( D(\alpha_n) )</th>
</tr>
</thead>
</table>
| 1 | 1 | \( 0, 1 \) | 2.360679 ... | 1.1708203 ...
| 2 | 2 | \( 0, 2 \) | 2.8284271 ... | 1.2071067 ...
| 3 | 5 | \( 0, 2, 12 \) | 2.9732137 ... | 1.2735737 ...
| 4 | 13 | \( 0, 2, 1, 4 \) | 2.9960526 ... | 1.2830816 ...
| 5 | 29 | \( 0, 2, 4, 12 \) | 2.9992071 ... | 1.2760671 ...
| 6 | 34 | \( 0, 2, 16, 1 \) | 2.9994232 ... | 1.2846465 ...
| 7 | 89 | \( 0, 2, 1, 18 \) | 2.9999158 ... | 1.2846662 ...
| 8 | 169 | \( 0, 2, 6, 12 \) | 2.9999766 ... | 1.2761401 ...
| 9 | 194 | \( 0, 2, 1, 2, 2, 14 \) | 2.9999822 ... | 1.2835097 ...
| 10 | 233 | \( 0, 2, 1, 10 \) | 2.9999877 ... | 1.2846956 ...
| 11 | 433 | \( 0, 2, 1, 2, 2, 12 \) | 2.9999964 ... | 1.2763673 ...
| 12 | 610 | \( 0, 2, 1, 12 \) | 2.9999982 ... | 1.2846990 ...
| 13 | 985 | \( 0, 2, 8, 12 \) | 2.9999993 ... | 1.2761423 ...
| 14 | 1325 | \( 0, 2, 1, 14, 2, 16 \) | 2.9999996 ... | 1.2845284 ...
| 15 | 1597 | \( 0, 2, 1, 14 \) | 2.9999997 ... | 1.2847005 ...
| 16 | 2897 | \( 0, 2, 1, 2, 2, 12, 2, 14 \) | 2.9999999 ... | 1.2835116 ...

Acknowledgement. The basic results of this work form a part of the author’s thesis done while at SUNY, Buffalo. I wish to thank Professor Thomas W. Cusick for his patient and invaluable guidance during this work, and for the reference [1] and [3], without which this
would not have been. I am also thankful to the referee for the innumerable suggestions that have made this work a little more readable and also for the list of counterexamples that appear at the end.

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Received on 11.5.1990
and in revised form on 14.12.1990 and 9.7.1992