GRAPHIC SEQUENCES OF TREES AND A PROBLEM OF FROBENIUS

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Abstract. We give a necessary and sufficient condition for the existence of a tree of order n with a given degree set. We relate this to a well-known linear Diophantine problem of Frobenius.

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Graphic sequences are sequences of positive integers which can be realized as the sequence of degrees of vertices of some simple graph. Graphic sequences have been much studied in the literature, and some of the most well-known characterizations of such sequences are given in [2], [4], [5]. Graphic sets are sets of positive integers which are formed by the degrees of vertices of some simple graph. Each set of positive integers is graphic, as is easy to see. For a given set S of positive integers, the least order of a graph for which S is its degree set was determined in [6]. This result has been generalized in [1] to the determination of all n for which there exist n-vertex graphs with degree set S.

We call a sequence of positive integers *tree-graphic* provided it can be realized as the degree sequence of a tree. The following result characterizes tree-graphic sequences. Although it is well-known, we give a proof for the sake of completeness.

Theorem 1. *Let $a_1, \ldots, a_n$ be positive integers, with $n \geqslant 2$. Then there exists a tree with degree sequence $a_1, \ldots, a_n$ if and only if $\sum\limits_{i=1}^{n} a_i = 2n - 2$.*

Proof. Suppose $s := a_1, \ldots, a_n$ is a sequence of positive integers with sum $2n - 2$. We show, by induction on the number of terms in the sequence, that there exists a tree with degree sequence s. The only 2-term sequence is 1, 1 and corresponds to the tree $\mathscr{K}_2$. Assuming the result for all such sequences with fewer than n terms, let $s := a_1, \ldots, a_n$ be a sequence, arranged in decreasing order. Then $a_{n-1} = a_n = 1$ and $a_1 > 1$. Consider the $(n-1)$-term sequence $s' := a_1 - 1, a_2, \ldots, a_{n-1}$ of positive integers with sum $2n - 4$. By induction hypothesis, let T' be a tree with degree sequence s'. The graph constructed by adding a new vertex to T' and adjoining it to the vertex of degree $a_1 - 1$ is a tree with degree sequence s. This completes the inductive proof. The converse is straightforward. □

An immediate and easy consequence of Theorem 1 is that every nontrivial tree must have at least two vertices of degree 1 since the average degree of a vertex in any tree is less than 2 and since there cannot be an isolated vertex. Therefore, in order to realize a set of positive integers as the degree set of a tree it is necessary that the set have least element 1. It is easy to determine the least order of a tree with a given degree set as a consequence of Theorem 1. Later on, we will also obtain Theorem 2 as a special case of Theorem 3.

Theorem 2. *Let S be a set of k positive integers with least element 1 whose elements sum to σ. Let $l_{\text{tree}}(S)$ be the least order of a tree with degree set S. Then $l_{\text{tree}}(S) = \sigma - k + 2$.*

Proof. Let any tree of least order have n vertices. There must be a unique vertex corresponding to each degree > 1, and hence $n - (k - 1)$ end-vertices. By Theorem 1, this implies $(\sigma - 1) + (n - k + 1) = 2n - 2$, whence $n = \sigma - k + 2$. □

Theorem 2 can be extended to the problem of characterizing n for which there is a tree with n vertices with a given degree set S.

Theorem 3. *Let $S = \{1, a_1, a_2, \ldots, a_{k-1}\}$, with $1 < a_1 < a_2 < \ldots < a_{k-1}$. Then there exists a tree of order n with degree set S if and only if the equation*

$$(a_1 - 1)x_1 + (a_2 - 1)x_2 + \ldots + (a_{k-1} - 1)x_{k-1} = n - 2 \tag{1}$$

is solvable in integers $x_i \geqslant 1$. If $S = \{1\}$, the only tree with degree set S is $\mathscr{K}_2$.

Proof. The case $S = \{1\}$ is easy. We henceforth assume that $|S| > 1$. Let T be a tree of order n, with degree set S. Let there be x_i vertices of degree a_i for $1 \leqslant i \leqslant k-1$, and let m denote the number of end-vertices. Then $m = n-(x_1+x_2+\ldots+x_{k-1})$, and by Theorem 1,

$$a_1x_1 + a_2x_2 + \ldots + a_{k-1}x_{k-1} = (2n - 2) - m = n - 2 + (x_1 + x_2 + \ldots + x_{k-1}).$$

Hence $\sum_{i=1}^{k-1} (a_i - 1)x_i = n - 2$, with each $x_i \geqslant 1$.

Conversely, suppose (1) has a solution in positive integers x_i. We construct a tree of order n with degree set $S = \{1, a_1, \ldots, a_{k-1}\}$. Observe that $\mathscr{K}_2$ is the only possibility when $S = \{1\}$. If $|S| > 1$, beginning with $T_1 = \mathscr{K}_{1,a_1}$, we adjoin $(a_1 - 1)$ new vertices to any one of the end-vertices to form T_2, and continue this process of adjoining (a_1-1) new vertices to any one of the end-vertices in T_i to obtain T_{i+1}. For each $i \geqslant 1$, each vertex in T_{i+1} not in T_i has degree 1, while the vertex in T_i with new vertices adjoined has degree a_1 in T_{i+1}. Moreover, T_{x_1} has $(a_1+1)+(a_1-1)(x_1-1) = 2 + (a_1 - 1)x_1$ vertices. We continue this process to obtain larger and larger trees adjoining (a_2-1) new vertices x_2 times, then adjoining (a_3-1) new vertices x_3 times, and so on, until we have adjoined $(a_{k-1} - 1)$ new vertices x_{k-1} times. The tree thus formed has $2+\sum\limits_{i=1}^{k-1}(a_i-1)x_i$ vertices and degree set S. This completes the constructive proof. □

It is easy to see that Theorem 2 follows immediately from Theorem 3 by choosing each $x_i = 1$. Theorem 3 shows that the problem of characterizing n for which there exist an n-vertex tree with degree set S is dependent on solving a linear Diophantine equation. A necessary and sufficient condition for the solvability of (1) in integers is that

$$d := \gcd(a_1 - 1, a_2 - 1, \ldots, a_{k-1} - 1) \mid (n - 2).$$

Since we seek solutions with each $x_i \geqslant 1$, the divisibility condition is only necessary. The case $k = 2$ is trivial; trees of order n exist if and only if $n \in (a_1 - 1)\mathbb{N} + 2$. For $k \geqslant 2$, we reduce (1) to the equivalent

$$b_1x_1 + b_2x_2 + \ldots + b_{k-1}x_{k-1} = \frac{n-2}{d}, \tag{2}$$

where $b_i = (a_i-1)/d$. There is no nice sufficient condition for the solvability of (2) in positive integers for $k > 3$, but it is not difficult to show that there is a least positive integer $f = f(b_1, b_2, \ldots, b_{k-1})$ such that (2) has a solution whenever the right-hand side of (2) exceeds f. Moreover, the least such positive integer in the case $k = 3$ is given by $f(b_1, b_2) = b_1b_2$. This is the much studied problem of Frobenius; for an extensive study, see [3].

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