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Note

A note on a theorem of Erdős & Gallai

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Abstract

We show that the Erdős–Gallai condition characterizing graphical degree sequences of length p needs to be checked only for as many n as there are distinct terms in the sequence, not all n , $1 \leq n \leq p$.

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A sequence d_1, d_2, \dots, d_p of nonnegative integers is called the *degree sequence* of a graph \mathcal{G} if the vertices of \mathcal{G} can be labeled v_1, v_2, \dots, v_p such that $\deg v_k = d_k$ for each k , $1 \leq k \leq p$. A sequence a_1, a_2, \dots, a_p of nonnegative integers is called *graphical* if it is the degree sequence of some graph. Any graphical sequence clearly satisfies the two conditions $a_k \leq p - 1$ for each k and $\sum_{k=1}^p a_k$ is *even*. However, these two conditions together do not ensure that a sequence will be graphical. Necessary and sufficient conditions for a sequence of nonnegative integers to be graphical are well known. Two characterizations of graphical sequences are due to Havel, Hakimi and Erdős and Gallai:

Theorem H–H (Havel [3] and Hakimi [2]). *A sequence of nonnegative integers a_1, a_2, \dots, a_p with $a_1 \geq a_2 \geq \dots \geq a_p$, $a_1 \geq 1$ and $p \geq 3$ is graphical if and only if the sequence $a_2 - 1, a_3 - 1, \dots, a_{a_1+1} - 1, a_{a_1+2}, \dots, a_p$ is graphical.*

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Theorem EG (Erdős and Gallai [1]). *A sequence of positive integers a_1, a_2, \dots, a_p with $a_1 \geq a_2 \geq \dots \geq a_p$ is graphical if and only if $\sum_{k=1}^p a_k$ is even and for each integer n , $1 \leq n \leq p-1$,*

$$\sum_{k=1}^n a_k \leq n(n-1) + \sum_{k=n+1}^p \min(n, a_k).$$

The theorem of Erdős and Gallai (Theorem EG) requires the verification of an inequality for each n , $1 \leq n \leq p-1$. It is interesting to observe that the inequality need not be checked for $n > s$, where s is the largest integer such that $a_s \geq s-1$. For $n > s$, the inequality in Theorem EG reduces to

$$\sum_{k=1}^n a_k \leq n(n-1) + \sum_{k=n+1}^p a_k.$$

Consider the difference between the right- and left-hand sides of the inequality as a function of n . Replacing n by $n+1$, this difference increases by $2(n-a_{n+1}) > 0$ since $n > s$. Thus, assuming the inequality holds for all $n \leq s$, it will also hold for all $n \leq p-1$. We record this observation as a

Lemma. *Let $\{a_1, a_2, \dots, a_p\}$ be a sequence of positive integers with $a_1 \geq a_2 \geq \dots \geq a_p$. Let s be the largest integer such that $a_s \geq s-1$. Then the sequence $\{a_1, a_2, \dots, a_p\}$ is graphical if and only if $\sum_{k=1}^p a_k$ is even and for each integer n , $1 \leq n \leq s$,*

$$\sum_{k=1}^n a_k \leq n(n-1) + \sum_{k=n+1}^p \min(n, a_k).$$

The lemma states that the number of inequalities to check in Theorem EG can be reduced. The purpose of this note is to prove a refined form of Theorem EG mentioned in the introduction. We show that in case of multiple occurrences of numbers in the degree sequence, it suffices to check the inequality in Theorem EG only at the end of each segment of repeated values. Throughout this paper we shall employ the notation $(a)_m$ to denote m occurrences of the integer a . Thus, we may denote a typical degree sequence by

$$s := (a_1)_{m_1}, (a_2)_{m_2}, \dots, (a_\ell)_{m_\ell}, \quad (1)$$

where $a_1 > a_2 > \dots > a_\ell$ and each $m_k \geq 1$ with $m_1 + m_2 + \dots + m_\ell = p$.

We shall write

$$\sigma_k := \sum_{i=1}^k m_i, \text{ with } \sigma_0 := 0, \text{ and } S_{r,t} := \sum_{i=r}^t a_i m_i.$$

Our main result is the

Theorem. *A sequence (1) is graphical if and only if $S_{1,\ell}$ is even and the inequality in Theorem EG holds for $n = \sigma_k$, $1 \leq k \leq \ell$.*

Proof. By Theorem EG, we only need to prove that checking the inequality at each σ_k implies the inequality holds at each n . Suppose the inequality holds at each σ_k , but is not valid for some $n = N$. Let N_0 be the *least* such N , and write

$$N_0 = \sigma_k + n', \text{ where } 1 \leq n' < m_{k+1} \text{ and } 0 \leq k < \ell.$$

Thus,

$$\begin{aligned} S_{1,k} + a_{k+1}n' &> (\sigma_k + n')(\sigma_k + n' - 1) + (m_{k+1} - n') \min(a_{k+1}, \sigma_k + n') \\ &\quad + m_{k+2} \min(a_{k+2}, \sigma_k + n') + \cdots + m_\ell \min(a_\ell, \sigma_k + n') \end{aligned} \tag{2}$$

and

$$\begin{aligned} S_{1,k} + a_{k+1}(n' - 1) &\leq (\sigma_k + n' - 1)(\sigma_k + n' - 2) + (m_{k+1} - n' + 1) \min(a_{k+1}, \sigma_k + n' - 1) \\ &\quad + m_{k+2} \min(a_{k+2}, \sigma_k + n' - 1) + \cdots + m_\ell \min(a_\ell, \sigma_k + n' - 1). \end{aligned} \tag{3}$$

Suppose now that $a_{k+1} < \sigma_k + n'$. Then subtracting (3) from (2) gives the inequality

$$a_{k+1} > 2(\sigma_k + n' - 1) - a_{k+1},$$

which contradicts our assumption. Thus,

$$a_{k+1} \geq \sigma_k + n' \tag{4}$$

and (2) reduces to

$$\begin{aligned} S_{1,k} + a_{k+1}n' &> (\sigma_k + n')(\sigma_{k+1} - 1) + m_{k+2} \min(a_{k+2}, \sigma_k + n') \\ &\quad + \cdots + m_\ell \min(a_\ell, \sigma_k + n'). \end{aligned} \tag{5}$$

Let r be such that $a_r < \sigma_k + n' \leq a_{r-1}$. Such an r exists because $\sigma_k + n' \leq a_\ell$ together with (5) would imply

$$a_1(\sigma_k + n') \geq S_{1,k} + a_{k+1}n' > (\sigma_k + n')(\sigma_\ell - 1),$$

which is impossible since $a_1 \leq p - 1 = \sigma_\ell - 1$. From (4), $r \geq k + 2$, and (5) further reduces to

$$S_{1,k} + a_{k+1}n' > (\sigma_k + n')(\sigma_{r-1} - 1) + S_{r,\ell} \tag{6}$$

and (3) similarly to

$$S_{1,k} + a_{k+1}(n' - 1) \leq (\sigma_k + n' - 1)(\sigma_{r-1} - 1) + S_{r,\ell} \tag{7}$$

Now subtracting (7) from (6) yields

$$a_{k+1} \geq \sigma_{r-1}. \tag{8}$$

Let n'' be the *largest* integer $\leq m_{k+1}$ for which the inequality in Theorem EG is not valid for $\sigma_k + n''$; from the definition, $n' \leq n'' < m_{k+1}$. Furthermore, analogous to (2)

and (3), we have

$$\begin{aligned} S_{1,k} + a_{k+1}n'' &> (\sigma_k + n'')(\sigma_k + n'' - 1) + (m_{k+1} - n'') \min(a_{k+1}, \sigma_k + n'') \\ &\quad + m_{k+2} \min(a_{k+2}, \sigma_k + n'') + \cdots + m_\ell \min(a_\ell, \sigma_k + n'') \end{aligned} \quad (9)$$

and

$$\begin{aligned} S_{1,k} + a_{k+1}(n'' + 1) &\leq (\sigma_k + n'')(\sigma_k + n'' + 1) + (m_{k+1} - n'' - 1) \min(a_{k+1}, \sigma_k + n'' + 1) \\ &\quad + m_{k+2} \min(a_{k+2}, \sigma_k + n'' + 1) + \cdots + m_\ell \min(a_\ell, \sigma_k + n'' + 1). \end{aligned} \quad (10)$$

By (8), $\sigma_k + n'' < \sigma_{k+1} \leq \sigma_{r-1} \leq a_{k+1}$. Define s such that $a_s \leq \sigma_k + n'' < a_{s-1}$, and note that $k + 1 < s \leq r$. Now, the difference between the right-hand sides of (9) and (10) equals

$$2(\sigma_k + n'') + (m_{k+1} - \sigma_k - 2n'' - 1) + m_{k+2} + \cdots + m_{s-1} = \sigma_{s-1} - 1.$$

Since a_{k+1} is the difference between the left-hand sides of (9) and (10), the inequalities

$$\sigma_{s-1} - 1 < \sigma_{s-1} \leq \sigma_{r-1} \leq a_{k+1},$$

lead to a contradiction. This completes the proof of our result. \square

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