
ELSEVIER
Volume 129, Issue 7, July 2009 ISSN 0022-314X
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Journal of Number Theory 129 (2009) 1805–1811

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the largest size of a partition that is both s-core and t-core

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ARTICLE INFO

Article history:
Received 28 June 2008
Revised 15 August 2008
Available online 20 March 2009
Communicated by M. Beck

MSC:
05A17
11P81
11P99

Keywords:
Partition
Hook
Hook number
t-Core
Structure number
t-Abacus

ABSTRACT

Text. Let s, t be relatively prime positive integers. We prove a conjecture of Aukerman, Kane and Sze regarding the largest size of a partition that is simultaneously s-core and t-core by solving an equivalent problem concerning sets S of positive integers with the property that for $n \in S$, $n - s \in S$ whenever $n \geqslant s$ and $n - t \in S$ whenever $n \geqslant t$.

Video. For a video summary of this paper, please visit http://www.youtube.com/watch?v=o1OEug8LryU.

1. Introduction

The theory of partitions traces its history back several centuries, to Euler and Jacobi, and [1] is a standard reference. In addition to a more apparent link to combinatorics, partitions of n are in one-to-one correspondence with irreducible representations of the symmetric group S_n; see [6] for a comprehensive account. Certain types of partitions, called t-core partitions, have played a role in combinatorial proofs of identities, and additionally have applications to Representation Theory when t is prime.

Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ denote a partition of size $|\Lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_m$, with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$. The *Ferrers–Young* diagram of Λ is a rectangular arrangement of nodes such that the ith row contains λ_i nodes. Corresponding to each node placed in the cell (i, j) is defined its *hook* which

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2008.08.009

consists of all nodes directly below or to the right of the node together with the node itself. The *hook number* $H(i,j)$ is the number of nodes in the hook of (i,j), and equals $(\lambda_i - i) + (\lambda'_j - j) + 1$, where λ'_j denotes the number of nodes in the jth column of the rectangular arrangement. For a positive integer t, a t-core partition is a partition Λ for which $t \nmid H(i,j)$ for each (i,j). The *structure numbers*, sometimes referred to as first column hook numbers, are the numbers $H(1,1), H(2,1), \ldots, H(m,1)$, that appear in the first column of the rectangular array of the hook numbers. Observe that the set S_Λ of structure numbers *uniquely* determine the partition Λ via $\lambda_i = H(i,1) - (m-i)$ for $1 \leqslant i \leqslant m$. Thus, we have

$$\sum_{n \in S_\Lambda} n = |\Lambda| + \frac{1}{2}|S_\Lambda|\big(|S_\Lambda| - 1\big). \tag{1}$$

Fix $t \in \mathbb{N}$. A *t-abacus* is a rectangular array with t columns in which a structure number (called a "bead") is placed in cell (r,c) provided it is of the form $rt + c$, $r \geqslant 0$, $0 \leqslant c < t$. Cells not occupied by the m beads are left blank. It is well known that a partition Λ is t-core if and only if each column of the corresponding t-abacus is filled by a bead in each of the first few rows, with no gap. Now, suppose Λ is t-core and $rt + c = n \in S_\Lambda$ occupies the cell (r,c) in the t-abacus. If $r = 0$, then $n = c < t$; if $r \geqslant 1$, then $n - t = (r-1)t + c$ occupies the cell $(r-1, c)$ in the t-abacus and must belong to S_Λ, since all cells above (r,c) in column c of the t-abacus must be occupied by elements of S_Λ. Conversely, suppose S is any set of positive integers such that $n \in S$ and $n \geqslant t$ together imply $n - t \in S$. If Λ is the partition whose structure set S_Λ is S, then Λ is t-core since each column in the t-abacus of Λ is occupied by elements of S_Λ in each of the first few rows, with no gap.

Let s,t be positive and relatively prime. The above argument, due to Kane and implicit in [2], demonstrates the connection between partitions Λ that are both s-core and t-core on one hand and sets S of positive integers (which turns out to be the set of its structure numbers S_Λ) which have the property that for $n \in S$, $n - s \in S$ if $n \geqslant s$ and $n - t \in S$ if $n \geqslant t$, on the other hand.

The study of partitions that are simultaneously s-core and t-core has been studied by several authors in the last decade; for example, [3,7,9]. Aukerman, Kane and Sze in [2, Conjecture 8.1] conjectured that the largest size of a partition that is both s-core and t-core is $(s^2-1)(t^2-1)/24$ if $\gcd(s,t) = 1$. Olsson and Stanton recently proved this conjecture in [8] by determining necessary and sufficient conditions for an s-block of integer partitions to be contained in a t-block. We give a simpler and much shorter proof of this conjecture by using the connection established above. More specifically, this together with (1), shows that conjecture is equivalent to proving that any set S of positive integers with the property that $n \in S$ implies $n - s \in S$ if $n \geqslant s$ and $n - t \in S$ if $n \geqslant t$ must satisfy

$$\sum_{n \in S} n \leqslant \frac{1}{2}|S|\big(|S| - 1\big) + \frac{1}{24}\big(s^2 - 1\big)\big(t^2 - 1\big), \tag{2}$$

and that $(s^2-1)(t^2-1)/24$ is the largest number in addition to $|S|(|S|-1)/2$ that serves as an upper bound in (2).

2. Proof of the conjecture

In this section, we prove (2). In fact, we also prove that equality in (2) appears precisely in the case $S = \{sx + ty \geqslant 1\colon\ x, y \in \mathbb{Z},\ xy < 0\}$, which we believe was conjectured by Kane and Sze in the 2004 West Coast Number Theory Conference held at UNLV, Las Vegas, NV, USA. The extremal set S is closely related to the Coin Exchange Problem of Frobenius, as we shall presently see.

Our proof of the conjecture relies on the correspondence between partitions that are simultaneously s-core and t-core and the sets of positive integers S which contains $n - s$ if $n \geqslant s$ and $n - t$ if $n \geqslant t$, whenever $n \in S$. Although we have indicated an argument for this in the Introduction, we use a different and more direct argument that does not rely on the concept of a t-abacus. As pointed out by one the referees, this result appears as [5, Lemma 1], but our proof of this is slightly different from that in [5] and we not only include it for this reason but also for the sake of completeness.

Theorem 1. *(Frame, Robinson and Thrall [5].) Let S_Λ denote the set of structure numbers of the partition $\Lambda := (\lambda_1, \lambda_2, \ldots, \lambda_m)$. Fix $H(r,1) \in S_\Lambda$. Then*

$$\{H(r,1) - H(i,1)\colon\ r+1 \leqslant i \leqslant m\} \cup \{H(r,j)\colon\ 2 \leqslant j \leqslant \lambda_r\} = \{1, 2, 3, \ldots, H(r,1) - 1\},$$

where the union is disjoint.

Proof. Fix r, where $1 \leqslant r \leqslant m$. We write $A := \{H(r,1) - H(i,1)\colon\ r+1 \leqslant i \leqslant m\}$, $B := \{H(r,j)\colon 2 \leqslant j \leqslant \lambda_r\}$ and $C := \{1, 2, 3, \ldots, H(r,1) - 1\}$ for convenience. If $r = m$ and $1 \leqslant j \leqslant \lambda_m$, we have $A = \emptyset$ and $H(m,j) = \lambda_m - j + 1$, so that $B = C$. Henceforth, we assume that $1 \leqslant r < m$. Since $A \cup B \subseteq C$ and $|A| + |B| = |C|$, it suffices to prove that $A \cap B = \emptyset$.

Consider $i \in \{r+1, r+2, \ldots, m-1\}$. We claim that

$$H(i,1) + H(r, \lambda_i + 1) < H(r,1) < H(i,1) + H(r, \lambda_i).$$

By definition, $\lambda'_{\lambda_i} \geqslant i$, with equality if and only if $\lambda_i > \lambda_{i+1}$. Hence

$$H(i,1) + H(r,\lambda_i) = \{(\lambda_i - i) + m\} + \{(\lambda_r - r) + (\lambda'_{\lambda_i} - \lambda_i) + 1\} \geqslant (\lambda_r - r) + m + 1 = H(r,1) + 1,$$

with equality if and only if $\lambda_i > \lambda_{i+1}$.

Again, by definition, $\lambda'_{\lambda_i+1} \leqslant i - 1$, with equality if and only if $\lambda_{i-1} > \lambda_i$. Hence

$$H(i,1) + H(r,\lambda_i + 1) = \{(\lambda_i - i) + m\} + \{(\lambda_r - r) + (\lambda'_{\lambda_i+1} - \lambda_i)\} \leqslant (\lambda_r - r) + m - 1 = H(r,1) - 1,$$

with equality if and only if $\lambda_{i-1} > \lambda_i$. Hence the claim, and the proof of the theorem. □

Corollary 1. *Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, and let $t \in \mathbb{N}$. Let S_Λ denote the set of structure numbers of Λ. If Λ is t-core, then S_Λ is a set of positive integers with the property that $n - t \in S_\Lambda$ whenever $n \in S_\Lambda$ and $n \geqslant t$. Conversely, if S is any set of positive integers such that $n \in S$ and $n \geqslant t$ together imply $n - t \in S$, then $S = S_\Lambda$ for some t-core partition Λ.*

Proof. Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, and let $t \in \mathbb{N}$. If Λ is t-core, $n \in S_\Lambda$ and $n \geqslant t$, then $n = H(r,1)$ for some r with $1 \leqslant r \leqslant m$. Since no hook number can equal t, by Theorem 1, $n - t = H(r',1)$ for some $r' > r$, so that $n - t \in S_\Lambda$. Conversely, suppose S is any set of positive integers with the property that $n \in S$ and $n \geqslant t$ together imply $n - t \in S$. The set S uniquely defines a partition Λ for which $S_\Lambda = S$. By Theorem 1, Λ is t-core, for if some $H(r,c) = \kappa t \in t\mathbb{N}$, then $n - \kappa t \notin S$ whereas $n = H(r,1) \in S$. □

Henceforth, we use the variables $a_1, a_2, \ldots, a_k$ in general or a, b when $k = 2$ in place of s, t.

Definition 1. Let $a_1, a_2, \ldots, a_k$ be positive integers with $\gcd(a_1, a_2, \ldots, a_k) = 1$. Let

$$\Gamma(a_1, a_2, \ldots, a_k) := \{a_1x_1 + a_2x_2 + \cdots + a_kx_k\colon\ x_i \in \mathbb{N} \cup \{0\}\},$$

and let $\Gamma^c(a_1, a_2, \ldots, a_k) := \mathbb{N} \setminus \Gamma(a_1, a_2, \ldots, a_k)$. Let m_i be the least positive integer in $\Gamma(a_2, a_3, \ldots, a_k)$ congruent to i modulo a_1 for $1 \leqslant i \leqslant a_1 - 1$.

It easily follows that $n \in \Gamma^c(a_1, a_2, \ldots, a_k)$ if and only if $n = m_i - a_1x_1$ for some $x_1 \in \mathbb{N}$ and some i, $1 \leqslant i \leqslant a_1 - 1$. In particular, $n \in \Gamma^c(a_1, a_2)$ if and only if $n = a_2 i - a_1x_1$ for some $x_1 \in \mathbb{N}$ and some i, $1 \leqslant i \leqslant a_1 - 1$. Henceforth, we write Γ and Γ^c to denote the sets $\Gamma(a_1, a_2, \ldots, a_k)$ and $\Gamma^c(a_1, a_2, \ldots, a_k)$, respectively, and write a, b for a_1, a_2, respectively.

Lemma 1. *Let $a_1, a_2, \ldots, a_k$ be positive integers with $\gcd(a_1, a_2, \ldots, a_k) = 1$. Let S be a set of positive integers such that for $n \in S$ and $1 \leqslant i \leqslant k$, $n \geqslant a_i$ implies $n - a_i \in S$. Then $S \subseteq \Gamma^c$, with equality if and only $m_i - a_1 \in S$ for $1 \leqslant i \leqslant k$.*

Proof. Suppose $n \in S \cap \Gamma$. Then $n = a_1x_1 + a_2x_2 + \cdots + a_kx_k$, so that $0 \in S$, which contradicts our assumption. Thus $S \subseteq \Gamma^c$. The conditions under which the equality holds follow easily from the remarks following Definition 1. □

Corollary 2. *Let a, b be positive integers with $\gcd(a, b) = 1$. Let S be a set of positive integers such that for $n \in S$, $n \geqslant a$ implies $n - a \in S$ and $n \geqslant b$ implies $n - b \in S$. Then $S \subseteq \Gamma^c$, with equality if and only $ab - a - b \in S$.*

Proof. Observe that the set of minima m_i is the set $\{bj\colon\ 1 \leqslant j \leqslant a-1\}$. The corollary now follows from Lemma 1 and the fact that $m_{-b} - a = b(a-1) - a \in S$ implies $m_{bi} - a = bi - a \in S$ for $1 \leqslant i \leqslant a-1$. □

The *Coin Exchange Problem* is the determination of the largest number in $\Gamma^c(a_1, a_2, \ldots, a_k)$, and was solved in the case $k = 2$ by Sylvester in [11]; this is the first part of the following theorem. Closely related is the second part, also due to Sylvester in [10]. The third part is due to Brown and Shiue in [4]; a shorter proof may be found in [12].

Theorem 2. *Let a, b be positive integers with $\gcd(a, b) = 1$. Then*

$$g(a,b) := \max \Gamma^c(a,b) = (a-1)(b-1) - 1;$$

$$n(a,b) := \left|\Gamma^c(a,b)\right| = \frac{1}{2}(a-1)(b-1);$$

$$s(a,b) := \sum_{n \in \Gamma^c(a,b)} n = \frac{1}{12}(a-1)(b-1)(2ab - a - b - 1).$$

Lemma 2. *Let a, b be positive integers with $\gcd(a, b) = 1$. Let S be any set of positive integers such that for $n \in S$, $n \geqslant a$ implies $n - a \in S$ and $n \geqslant b$ implies $n - b \in S$. For each i, $1 \leqslant i \leqslant a-1$, let λ_i be the least positive integer such that $bi - a\lambda_i \in S$. Then $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_{a-1}$.*

Proof. Suppose $1 \leqslant i < j \leqslant a-1$. From $bj - a\lambda_j \in S$ and $j > i$, it follows that $bi - a\lambda_j \in S$. Thus $\lambda_i \leqslant \lambda_j$ from the definition of λ_i. □

The following inequality is crucial to proving the main result.

Lemma 3. *Let a, b, t be positive integers, with $1 < a < b$. If $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{a-1}$, $t_i < bi/a$ for $1 \leqslant i \leqslant a-1$, and $t_1 + t_2 + \cdots + t_{a-1} = t$, then*

$$2b\sum_{i=1}^{a-1} it_i - a\sum_{i=1}^{a-1} t_i^2 > (a-1)bt - t^2. \tag{3}$$

Proof. We prove the result by inducting on a, for fixed b. For $a = 2$, (3) amounts to proving $2bt_1 - 2t_1^2 > bt_1 - t_1^2$, or $t_1 < b$, which is true by assumption. Assume the result for all values of $a \leqslant k$, where $k < b$. Let $a = k + 1 < b$, and consider any non-decreasing sequence of nonnegative integers $\{t_i\}_{i=1}^k$, where $t_i < bi/(k+1)$ for $1 \leqslant i \leqslant k$ and $\sum_{i=1}^k t_i = t$. Observe that the sequence $\{t_i\}_{i=1}^{k-1}$ satisfies $t_i < bi/k$ for $1 \leqslant i \leqslant k-1$ and $\sum_{i=1}^{k-1} t_i = t - t_k$. Hence, by induction hypothesis,

$$2b\sum_{i=1}^{k} it_i-(k+1)\sum_{i=1}^{k} t_i^2=\left(2b\sum_{i=1}^{k-1} it_i-k\sum_{i=1}^{k-1} t_i^2\right)+2bkt_k-(k+1)t_k^2-\sum_{i=1}^{k-1} t_i^2$$

$$> \left\{(k-1)b(t-t_k)-(t-t_k)^2\right\}+2bkt_k-(k+1)t_k^2-\sum_{i=1}^{k-1} t_i^2$$

$$=\left(kbt-t^2\right)-b(t-t_k)+bkt_k+2tt_k-(k+2)t_k^2-\sum_{i=1}^{k-1} t_i^2$$

$$=\left(kbt-t^2\right)-b(t-t_k)+\left\{(k-1)bt_k+bt_k\right\}$$

$$+\left\{2t_k(t-t_k)-(k-1)t_k^2-t_k^2\right\}-\sum_{i=1}^{k-1} t_i^2$$

$$=\left(kbt-t^2\right)+bt_k+b\sum_{i=1}^{k-1}(t_k-t_i)-t_k^2-\sum_{i=1}^{k-1}(t_k-t_i)^2$$

$$=\left(kbt-t^2\right)+(b-t_k)t_k+\sum_{i=1}^{k-1}(t_k-t_i)(b-t_k+t_i)$$

$$> kbt-t^2.$$

This completes the inductive proof. □

Remark 1. Write

$$S:=\sum_{i=1}^{a-1}(2bi-at_i)t_i=\frac{1}{a}\sum_{i=1}^{a-1}(bi)^2-\frac{1}{a}\sum_{i=1}^{a-1}(bi-at_i)^2=\frac{1}{6}(a-1)(2a-1)b^2-\frac{1}{a}T,$$

where $T:=\sum_{i=1}^{a-1}(bi-at_i)^2$. Observe that $bi-at_i$ are positive integers from the nonzero residue classes mod a with fixed sum $\frac{1}{2}a(a-1)b-at=a\{\frac{1}{2}(a-1)b-t\}=a\sigma$. A simple algebraic calculation shows that the inequality of Lemma 3 is equivalent to proving that

$$T:=\sum_{i=1}^{a-1}(bi-at_i)^2<a\left\{\sigma^2+\frac{1}{12}\left(a^2-1\right)b^2\right\}.$$

Theorem 3. *Let a, b be positive integers with $\gcd(a,b)=1$. Let S be any set of positive integers such that for $n\in S$,*

$$n\geqslant a \quad \textit{implies} \quad n-a\in S \quad \textit{and} \quad n\geqslant b \quad \textit{implies} \quad n-b\in S.$$

Then

$$\sum_{n\in S} n\leqslant \frac{1}{2}|S|\left(|S|-1\right)+\frac{1}{24}\left(a^2-1\right)\left(b^2-1\right), \tag{4}$$

with equality if and only if $ab-a-b\in S$.

Proof. From Corollary 1, $S \subseteq \Gamma^c$, with equality if and only if $ab - a - b \in S$. We first show that equality in (4) occurs when $S = \Gamma^c$. This translates to showing that

$$s(a,b) = \frac{1}{2}n(a,b)\{n(a,b) - 1\} + \frac{1}{24}(a^2 - 1)(b^2 - 1),$$

which can be easily verified by using the results of Theorem 2. Indeed,

$$\begin{aligned}
&\frac{1}{2}n(a,b)\{n(a,b) - 1\} + \frac{1}{24}(a^2 - 1)(b^2 - 1) \\
&\quad = \frac{1}{8}(a-1)(b-1)(ab-a-b-1) + \frac{1}{24}(a^2 - 1)(b^2 - 1) \\
&\quad = \frac{1}{24}(a-1)(b-1)\{3(ab-a-b-1) + (a+1)(b+1)\} \\
&\quad = \frac{1}{12}(a-1)(b-1)(2ab-a-b-1) \\
&\quad = s(a,b).
\end{aligned}$$

It remains to show that there is strict inequality in (4) when $ab - a - b \notin S$, or equivalently when $S \subset \Gamma^c$. Consider such a set S, and let $\Gamma^c \setminus S$ have t_i elements in the class bi mod a for $1 \leqslant i \leqslant a-1$. These elements must be $\{bi - ax_i\colon\ 1 \leqslant x_i \leqslant t_i\}$, with $t_1 \leqslant t_2 \leqslant t_3 \leqslant \cdots \leqslant t_{a-1}$ and $t_i < bi/a$ for $1 \leqslant i \leqslant a-1$, by Lemma 2. Set $t := t_1 + t_2 + \cdots + t_{a-1}$. We have equal sums on both sides of (4) when $S = \Gamma^c$. Removal of the t elements from Γ^c results in a *decrease* in the sum on the left-hand side of (4) by

$$\sum_{i=1}^{a-1}\left(bit_i - \frac{1}{2}a\,t_i(t_i+1)\right) = b\sum_{i=1}^{a-1} it_i - \frac{1}{2}a\sum_{i=1}^{a-1} t_i^2 - \frac{1}{2}at, \tag{5}$$

and a *decrease* in the sum on the right-hand side of (4) by

$$\frac{1}{2}n(n-1) - \frac{1}{2}(n-t)(n-t-1) = nt - \frac{1}{2}t(t+1) = \frac{1}{2}(a-1)(b-1)t - \frac{1}{2}t(t+1), \tag{6}$$

where $n = |\Gamma^c| = n(a,b) = \frac{1}{2}(a-1)(b-1)$. Thus, proving the strict inequality in (4) is equivalent to proving that the sum in (5) is greater than the sum in (6) when $1 \leqslant t \leqslant n-1$. This simplifies to

$$2b\sum_{i=1}^{a-1} it_i - a\sum_{i=1}^{a-1} t_i^2 > (a-1)bt - t^2, \tag{7}$$

which is the result of Lemma 3. Hence the theorem. □

We restate Theorem 3 in the equivalent form of the conjecture as follows:

Theorem 4. *The largest size of a partition that is both a-core and b-core is $(a^2-1)(b^2-1)/24$ if $\gcd(a,b) = 1$. Moreover, the unique partition λ of largest size has structure set $S_\Lambda = \{ax + by\colon\ x, y \in \mathbb{Z},\ xy < 0\} = \Gamma^c(a,b)$.*

Acknowledgments

The author is grateful to Gerry Myerson for introducing him to the conjecture, to Ben Kane for introducing him to the related problem, and to B.J. Venkatachala for helping in the proof of Lemma 3. The author also gratefully acknowledges the careful comments of the two referees.

Supplementary material

The online version of this article contains additional supplementary material.
Please visit doi:10.1016/j.jnt.2008.08.009.

References

[1] G.E. Andrews, The Theory of Partitions, Cambridge University Press, 1998.
[2] D. Aukerman, B. Kane, L. Sze, On simultaneous s-cores/t-cores, preprint.
[3] J.K. Anderson, Partitions which are simultaneously t_1- and t_2-core, Discrete Math. 248 (2002) 237–243.
[4] T.C. Brown, P.J. Shiue, A remark related to the Frobenius Problem, Fibonacci Quart. 31 (1993) 31–36.
[5] J.S. Frame, G. de B. Robinson, R.M. Thrall, The hook graphs of the symmetric group, Canad. J. Math. 6 (1954) 316–324.
[6] G. James, A. Kerber, The Representation Theory of the Symmetric Group, Encyclopedia Math. Appl., vol. 16, Addison–Wesley, 1981.
[7] G. Navarro, W. Willems, When is a p-block a q-block? Proc. Amer. Math. Soc. 125 (1997) 1589–1591.
[8] J.B. Olsson, D. Stanton, Block inclusions and cores of partitions, Aequationes Math. 74 (2007) 90–110.
[9] J.-C. Puchta, Partitions which are p- and q-core, Integers 1 (2001), A6, 3 pp.
[10] J.J. Sylvester, On subvariants, i.e. semi-invariants to binary quantitites of an unlimited order, Amer. J. Math. 5 (1882) 119–136.
[11] J.J. Sylvester, Mathematical questions with their solutions, Educational Times 41 (1884) 21.
[12] A. Tripathi, On sums of positive integers that are not of the form $ax+by=n$, Amer. Math. Monthly 115 (4) (2008) 363–364.