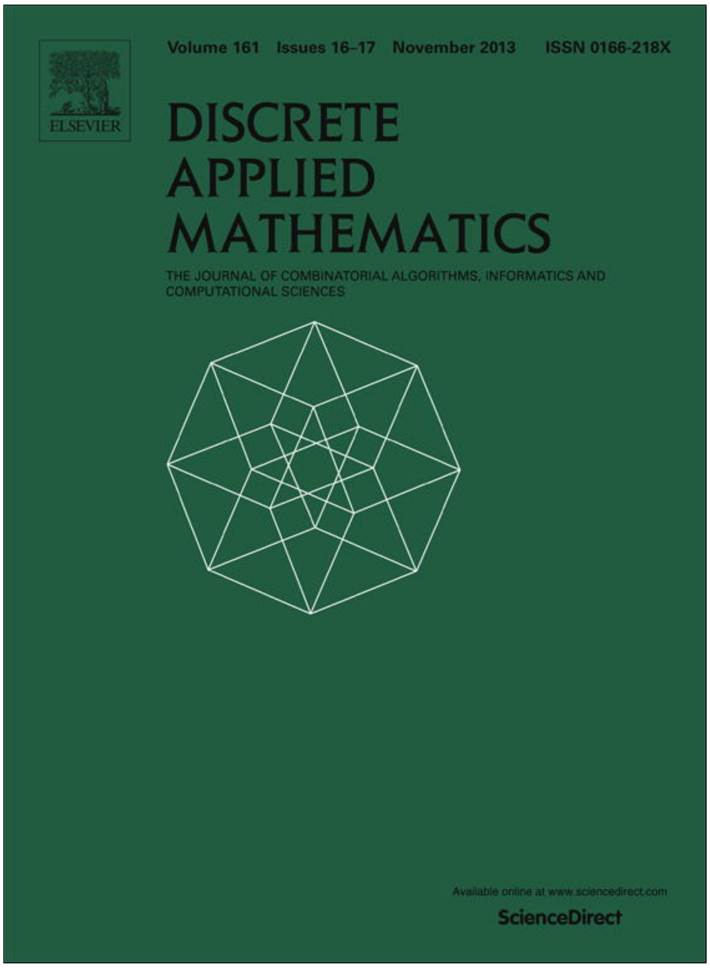
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On the range of size of sum graphs & integral sum graphs of a given order

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ABSTRACT

A finite simple graph G is called a `sum graph` (respectively, `integral sum graph`) if there is a *bijection* f from the vertices of G to a set of positive integers S (respectively, a set of integers S) such that uv is an edge of G if and only if $f(u)+f(v) \in S$. For graphs with n vertices, we show that there exist sum graphs with m edges if and only if $m \leq \lfloor \frac{1}{4}(n-1)^2 \rfloor$ and that there exists integral sum graphs with m edges if and only if $m \leq \lceil \frac{3}{8}(n-1)^2 \rceil + \lfloor \frac{1}{2}(n-1) \rfloor$, except for $m = \lceil \frac{3}{8}(n-1)^2 \rceil + \lfloor \frac{1}{2}(n-1) \rfloor - 1$ when n is of the form $4k+1$. We also characterize sets of positive integers (respectively, integers) which are in bijection with sum graphs (respectively, integral sum graphs) of maximum size for a given order.

1. Introduction

The notion of sum graph was introduced by Harary [8] in 1990. A graph $G(V, E)$ is called a `sum graph` if there is a *bijection* f from $V(G)$ to a set of positive integers S such that $uv \in E(G)$ if and only if $f(u)+f(v) \in S$. We call S a set of `labels` for the sum graph G, and denote this set by $\mathcal{L}(G)$. Conversely, any set of positive integers S induces a sum graph G_S with vertex set S and edges $s_i s_j$ whenever $s_i + s_j \in S$. Thus every sum graph can be realized as one induced by a (finite) set of positive integers. Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain isolated vertices. For a *connected* graph G, the `sum number` of G, denoted by $\sigma(G)$, is the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph. The sum number of various classes of graphs are known, among them $\mathcal{K}_n$, $\mathcal{K}_{m,n}$, $\mathcal{C}_n$ and trees [6]. In Section 2, we determine the range of size of sum graphs of a given order n, and also characterize the set of positive integers that induce sum graphs of maximum size for a given order.

Various extensions to the notion of sum graphs were introduced by several authors, among them integral sum graphs by Harary [9] to any finite set of integers in 1994, real sum graphs by Harary et al. [10] to any finite set of positive real numbers in 1991, and product graphs by Bergstrand et al. [2] to any finite set of positive integers not containing 1 and closed under the product in 1992. However, it turned out that every real sum graph is a sum graph (see [10]), and the set of all product graphs is identical to the set of all sum graphs (see [2]). On the other hand, the notion of `integral sum graphs` gave rise to a different class of graphs. Integral sum graphs arise from labelling the vertices from the set of integers, leading to an analogous definition of `integral sum number`, denoted by $\zeta(G)$. Unlike sum graphs, integral sum graphs need not have isolated vertices, and so $\zeta(G)$ may be 0. Much less is known about integral sum graphs than about sum graphs; see [6]. In Section 3, we determine the range of size of integral sum graphs of a given order n, and also characterize the set of integers that induce integral sum graphs of maximum size for a given order. In particular, we answer in the negative a conjecture of Nicholas & Vilfred [19] that there exists an integral sum graph of order n and size m if and only if $m \leq \frac{3}{8}(n^2-1) - \lfloor \frac{1}{4}(n-1) \rfloor$ when n is odd and $m \leq \frac{1}{8}n(3n-2)$ when n is even (there is a misprint in ([6], pp. 147)). In fact, we show the nonexistence

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of integral sum graphs of size $\frac{3}{8}(n^2-1)-\frac{1}{4}(n-1)-1$ when n is of the form $4k+1$, and confirm the existence of all other integral sum graphs in the conjecture.

The study of sum graphs has many applications. Slamet et al. [22] show how one can use sum graph labellings to distribute secret information to a set of people so that only an authorized set of people can reconstruct the secret (see [6], pp. 144). Sutton [23], in his Ph.D. thesis, introduced two methods of graph labellings that generalize the notion of sum graphs and have applications to storage and manipulation of relational database (see [6], pp. 149). The focus of research in the subject of sum graphs and integral sum graphs has been on determining the sum number [1,4,5,7,8,11,12,14,17,18,20,24,25] or integral sum number [3,9,13–16,21,25,26] of classes of graphs. However, it is a natural question to ask for the existence of sum and integral sum graphs of given order and given size, which has also been independently raised in [19] for integral sum graphs.

2. Range of size of sum graphs

An (n, m)-`sum graph` is a sum graph of order n and size m. In this section, for a given positive integer n, we determine all m for which there exists an (n, m)-sum graph. We also characterize sets of positive integers that induce a sum graph of maximum size among those of order n.

Lemma 2.1. *Any (n, m)-sum graph can be extended to a $(n+1, m)$-sum graph.*

Proof. Let G be an (n, m)-sum graph, induced by the set of positive integers $\mathcal{L}(G)$ with largest element ℓ. Then the sum graph G' induced by the set $\mathcal{L}(G) \cup \{2\ell\}$ is a $(n+1, m)$-sum graph, since the additional vertex in G' is isolated and any two non-adjacent vertices in G are also non-adjacent in G'. □

Theorem 2.1. *The maximum size for a sum graph of order n is $m_n = \lfloor \frac{1}{4}(n-1)^2 \rfloor$. Moreover, there exists a sum graph of order n and size m, for each $m \le m_n$.*

Proof. Throughout this proof we write $m_n = \lfloor \frac{1}{4}(n-1)^2 \rfloor$. There can be no edge in a sum graph of order 1 or 2. The sum graph induced by $\{1, 2, 3\}$ has one edge, while that induced by $\{1, 2, 4\}$ has no edge, verifying the result for $n = 3$. The sum graph induced by $\{1, 2, 3, 4\}$ has two edges, that induced by $\{1, 2, 3, 6\}$ has one edge, while that induced by $\{1, 2, 4, 7\}$ has no edge, verifying the result for $n = 4$. We may therefore assume that $n > 4$.

I. Let G be a sum graph of order n. Let $V(G) = \{v_1, \dots, v_n\}$, with labelling $\ell(v_i) = \ell_i$ for $1 \le i \le n$ and $\ell_1 < \cdots < \ell_n$. Let $\mathcal{L}(G) = \{\ell_1, \dots, \ell_n\}$. Thus $v_i \leftrightarrow v_j$ if and only if $\ell_i + \ell_j \in \mathcal{L}(G), j \ne i$. So, for each $k \in \{1, \dots, n\}$, the number of edges in G is *at most* the number of pairs $\{i, j\}$ such that $i + j = k$ with $i < j$. Thus the size of G is *at most*

$$\sum_{k=1}^{n} \left\lfloor \frac{1}{2}(k-1) \right\rfloor = \left\lfloor \frac{1}{4}(n-1)^2 \right\rfloor.$$

II. Consider the sum graph $\mathscr{G}_n$ with vertices $v_1, \dots, v_n$, induced by the set of integers in $[1, n]$. Label the vertices by $\ell(v_i) = i$ for $1 \le i \le n$, so that $v_i \leftrightarrow v_j$ if and only if $i + j \le n$ for $i \ne j$.

If $i \le \lfloor \frac{1}{2}n \rfloor$, then $N(v_i) = \{v_1, \dots, v_{n-i}\} \setminus \{v_i\}$ and $d(v_i) = n - i - 1$; if $i > \lfloor \frac{1}{2}n \rfloor$, then $N(v_i) = \{v_1, \dots, v_{n-i}\}$ and $d(v_i) = n - i$. Hence the number of edges in $\mathscr{G}_n$ equals

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{\lfloor n/2 \rfloor}(n-i-1) + \frac{1}{2}\sum_{i=\lfloor n/2 \rfloor+1}^{n}(n-i) &= \frac{1}{2}\left(\sum_{i=1}^{n-1}(n-i) - \sum_{i=1}^{\lfloor n/2 \rfloor} 1\right) \\ &= \frac{1}{2}\left(\frac{1}{2}n(n-1) - \left\lfloor \frac{1}{2}n \right\rfloor\right) \\ &= \left\lfloor \frac{1}{4}(n-1)^2 \right\rfloor.\end{aligned}$$

Hence $\mathscr{G}_n$ is a sum graph of order n and size $m_n = \lfloor \frac{1}{4}(n-1)^2 \rfloor$.

III. For $n > 4$ and $k \in [1, \lfloor \frac{1}{2}(n-1) \rfloor - 1]$, we construct a sum graph $G_{n,k}$ of order n and size $m_{n-1} + k = \lfloor \frac{1}{4}(n-2)^2 \rfloor + k = |E(\mathscr{G}_{n-1})| + k$. As k varies over the set $\{1, \dots, \lfloor \frac{1}{2}(n-1) \rfloor - 1\}$, this covers all integers in the interval $[m_{n-1}+1, m_n - 1]$, since $m_n - m_{n-1} = \lfloor \frac{1}{4}(n-1)^2 \rfloor - \lfloor \frac{1}{4}(n-2)^2 \rfloor = \lfloor \frac{1}{2}(n-1) \rfloor$. So if there is a $(n-1, m)$-sum graph for each $m \le m_{n-1}$, Lemma 2.1 ensures the existence of an (n, m)-sum graph for each $m \le m_{n-1}$, and hence for each $m \le m_n$, completing the construction by induction on n.

Consider the sum graph $G_{n,k}$ with vertices $v_1, \dots, v_n$, induced by the set of integers in $[1, n-1] \cup \{x\}$, where $x \ge n$ will be specified later. Label the vertices by $\ell(v_i) = i$ for $1 \le i \le n-1$ and $\ell(v_n) = x$. The subgraph with vertices in $\{v_1, \dots, v_{n-1}\}$ and the edges v_iv_j whenever $i + j \le n - 1$ is the sum graph $\mathscr{G}_{n-1}$; this has size $\lfloor \frac{1}{4}(n-2)^2 \rfloor$. The only other edges in $G_{n,k}$ are v_iv_j with $i + j = x$. To count these, assume $i < j$. Then $i \in \{x - n + 1, \dots, \lfloor \frac{1}{2}(x-1) \rfloor\}$, since $i < j \le n - 1$. So the number of edges v_iv_j equals $n - \lceil \frac{1}{2}(x+1) \rceil$, and this equals k exactly when $x \in \{2(n-k-1), 2(n-k)-1\}$. Note that $2(n-k-1) \ge n$ if and only if $k \le \frac{1}{2}n - 1$. Hence the number of edges in $G_{n,k}$ equals $m_{n-1} + k$ when $G_{n,k}$ is induced by the set of integers in $[1, n-1] \cup \{2(n-k-1)\}$ or the set of integers in $[1, n-1] \cup \{2(n-k)-1\}$. This completes the proof. □

Remark 2.1. The maximum size in Theorem 2.1 reduces to $\frac{1}{4}(n-1)^2$ when n is odd and to $\frac{1}{4}n(n-2)$ when n is even.

We close this section by determining those sets of positive integers S which induce an (n, m_n)-sum graph. The construction in Theorem 2.1 establishes that $S = \{1, \dots, n\}$ is one such set, for each $n \geq 1$. Since the graphs induced by the sets S and cS are isomorphic for any $c \geq 1$ and any set S of positive integers, it suffices to consider only those sets S for which $\gcd(S) = 1$; we call these sets `reduced`. For $n \geq 5$, we show that the only reduced set which induces an (n, m_n)-sum graph is $\{1, \dots, n\}$. There are a few additional reduced sets when $n \leq 4$.

Theorem 2.2. *Let S be a reduced set of positive integers such that $|S| = n$ and $|E(G_S)| = m_n = \lfloor \frac{1}{4}(n-1)^2 \rfloor$.*

(i) *If $n \leq 4$, the set S is given by*

$$\begin{cases} \{1\} & \text{if } n = 1; \\ \{a, b\} & \text{if } n = 2; \\ \{a, b, a+b\} & \text{if } n = 3; \\ \{a, b, a+b, 2a+b\} \text{ or } \{a, b, a+b, a+2b\} & \text{if } n = 4, \end{cases}$$

with $\gcd(a, b) = 1$.

(ii) *If $n \geq 5$,*

$$S = \begin{cases} \{1, \dots, n\} & \text{if } n \text{ is odd}; \\ \{1, \dots, n\} \text{ or } \{1, \dots, n-1, n+1\} & \text{if } n \text{ is even}. \end{cases}$$

Proof. Throughout this proof, we write $m_n = \lfloor \frac{1}{4}(n-1)^2 \rfloor$.

(i) The cases where $n = 1$ or 2 are trivial.

Suppose $S = \{a, b, c\}$ with $a < b < c$, and $G_S = \{v_1, v_2, v_3\}$ with $\ell(v_1) = a$, $\ell(v_2) = b$ and $\ell(v_3) = c$. Since $m_3 = 1$ and $d(v_3) = 0$, $v_1 \leftrightarrow v_2$, so that $a + b = c$. Thus $S = \{a, b, a+b\}$ when $n = 3$.

Finally suppose $S = \{a, b, c, d\}$ with $a < b < c < d$, and $G_S = \{v_1, v_2, v_3, v_4\}$ with $\ell(v_1) = a$, $\ell(v_2) = b$, $\ell(v_3) = c$ and $\ell(v_4) = d$. Since $m_4 = 2$ and $d(v_4) = 0$, there are three possibilities: (i) $v_1 \not\leftrightarrow v_2$; (ii) $v_1 \not\leftrightarrow v_3$; and (iii) $v_2 \not\leftrightarrow v_3$. In case (i), v_3 is adjacent to both v_1 and v_2; this is impossible since both $c + a$ and $c + b$ must then equal d. In case (ii), v_2 is adjacent to both v_1 and v_3. Thus both $b+a$ and $b+c$ must belong to $\{a, b, c, d\}$, which can only happen if $b+c = d$ and $b+a = c$; hence $S = \{a, b, a+b, a+2b\}$. In case (iii), v_1 is adjacent to both v_2 and v_3. Thus both $a + b$ and $a + c$ must belong to $\{a, b, c, d\}$, which can only happen if $a + c = d$ and $a + b = c$; hence $S = \{a, b, a+b, 2a+b\}$. Therefore $S = \{a, b, a+b, 2a+b\}$ or $\{a, b, a+b, a+2b\}$ when $n = 4$.

(ii) We prove the result for $n \geq 5$ by induction. We note that if a sum graph has vertices $v_1, \dots, v_n$ listed in increasing order of labelling, then $d(v_i) \leq n - i$ for $1 \leq i \leq n$ since there are only $n - i$ vertices with a higher label than v_i.

Suppose $S = \{a, b, c, d, e\}$ with $a < b < c < d < e$, and $G_S = \{v_1, v_2, v_3, v_4, v_5\}$ with $\ell(v_1) = a$, $\ell(v_2) = b$, $\ell(v_3) = c$, $\ell(v_4) = d$ and $\ell(v_5) = e$. Note that $m_5 = 4$ and $d(v_5) = 0$. Therefore none of the other four vertices can be isolated, and since $d(v_4) \leq 1$, we must have $d(v_4) = 1$. Since 3, 3, 1, 1 is not a graphic sequence and $d(v_3) \leq 2$, we now have $d(v_3) = 2$. Thus $\{d(v_1), d(v_2)\} = \{2, 3\}$.

If $d(v_2) = 3$, then $N(v_2) = \{v_1, v_3, v_4\}$; so $(b + a, b + c, b + d) = (c, d, e)$. The only other edge is v_1v_3, and this gives $a + c = e$ (since $a + c \neq d = b + c$). Putting all this together, we arrive at the contradiction $a + c = e = b + d = 2b + c$. Thus $d(v_1) = 3$ and the edges are $v_1v_2, v_1v_3, v_1v_4, v_2v_3$. Thus $(a + b, a + c, a + d) = (c, d, e)$ and so $b + c = e$, since $b + c \neq d = a + c$. Thus $S = \{a, b, a+b, 2a+b, 3a+b\}$ with $a + 2b = 3a + b$. But then $b = 2a$, so that $a = 1$, $b = 2$ and $S = \{1, \dots, 5\}$. This completes the assertion for the base case $n = 5$.

Suppose $S = \{a_1, \dots, a_n\}$, arranged in increasing order, induces the sum graph $G_S = \{v_1, \dots, v_n\}$ with $\ell(v_i) = a_i$ for $1 \leq i \leq n$ of size m_n for some $n > 5$. Note that $d(v_n) = 0$. Write $S' = S \setminus \{a_n\}$. Now $|E(G_S)| - |E(G_{S'})|$ equals the number of edges v_iv_j for which $a_i + a_j = a_n$. Since the number of such edges cannot exceed $\lfloor \frac{1}{2}(n-1) \rfloor$ and $|E(G_S)| = m_n$, we have $|E(G_{S'})| \geq m_n - \lfloor \frac{1}{2}(n-1) \rfloor = m_{n-1}$. It follows that $|E(G_{S'})| = m_{n-1}$, so that $S' = \{a_1, \dots, a_{n-1}\}$ also induces a sum graph of maximum size.

If n is even, $S' = \{1, \dots, n-1\}$ by the induction hypothesis. We claim that in order for $S = S' \cup \{a_n\}$ to induce a sum graph of size m_n, $a_n = n$ or $n + 1$. Since $d(v_n) = 0$ in G_S, the additional $\frac{1}{2}n - 1$ edges are v_iv_j for $1 \leq i < j \leq n - 1$ with $i + j = a_n$. Thus $i \in \{a_n - n + 1, \dots, \lfloor \frac{1}{2}(a_n - 1) \rfloor\}$, since $a_n \geq n$, and the number of edges thus equals $n - 1 - \lfloor \frac{1}{2}a_n \rfloor$. Since this must equal $\frac{1}{2}n - 1$, we have $a_n = n$ or $n + 1$, as desired.

If n is odd, $S' = \{1, \dots, n-1\}$ or $\{1, \dots, n-2, n\}$ by the induction hypothesis. We claim that in order for $S = S' \cup \{a_n\}$ to induce a sum graph of size m_n, we must have $a_n = n$ in the first case and $a_n = n - 1$ in the second case. In the first case, $d(v_n) = 0$ in G_S and the additional $\frac{1}{2}(n-1)$ edges are v_iv_j for $1 \leq i < j \leq n - 1$ with $i + j = a_n$. As in the case when n is even, this implies $n - 1 - \lfloor \frac{1}{2}a_n \rfloor = \frac{1}{2}(n-1)$. This gives $a_n = n$, since $a_n > n - 1$, as desired. In the second case, adding $a_n = n - 1$ induces the sum graph G_S with m_n edges. If $a_n > n$, $d(v_n) = 0$ in G_S and the additional $\frac{1}{2}(n-1)$ edges are v_iv_j for $1 \leq i < j \leq n - 2$ or $j = n$, with $i + j = a_n$. Thus $i \in \{a_n - n, a_n - n + 2, \dots, \lfloor \frac{1}{2}(a_n - 1) \rfloor\}$, and the number of edges again equals $n - 1 - \lfloor \frac{1}{2}a_n \rfloor$. Since this must equal $\frac{1}{2}(n-1)$, we have $a_n = n - 1$, which is impossible.

This completes the proof by induction. □

3. Range of size of integral sum graphs

An (n, m)-integral sum graph is an integral sum graph of order n and size m. In this section, for a given positive integer n, we determine all m for which there exists an (n, m)-integral sum graph. Since sum graphs are automatically integral sum graphs, the range of m includes all nonnegative integers less than or equal to $\lfloor \frac{1}{4}(n-1)^2 \rfloor$ by Theorem 2.1. We also characterize sets of integers that induce an integral sum graph of maximum size among those of order n.

Let $n \geq 1$. The function

$$f(r, s) = m_r + m_s + rs = \left\lfloor \frac{1}{4}(r-1)^2 \right\rfloor + \left\lfloor \frac{1}{4}(s-1)^2 \right\rfloor + rs \tag{1}$$

with $r, s \geq 0$ is crucial to determining the maximum size of integral sum graphs. Since $m_n - m_{n-1} = \lfloor \frac{1}{2}(n-1) \rfloor$, we have

$$\begin{aligned} f(r-1, s+1) - f(r, s) &= \left(m_{s+1} - m_s\right) - \left(m_r - m_{r-1}\right) + r - s - 1 \\ &= \left(r - 1 - \left\lfloor \frac{1}{2}(r-1) \right\rfloor\right) - \left(s - \left\lfloor \frac{1}{2}s \right\rfloor\right) \\ &= \left\lceil \frac{1}{2}(r-1) \right\rceil - \left\lceil \frac{1}{2}s \right\rceil. \end{aligned} \tag{2}$$

So if $r - s > 1$ and $r + s$ is a fixed positive integer, then $f(r-1, s+1) - f(r, s) \geq 0$. Thus $f(r, s)$ attains its maximum when $|r - s| \leq 1$ if $r + s$ is fixed.

We show that the maximum value that $f(r, s)$ attains with $r + s = n - 1$ is given by $M_n - (n-1)$, where

$$M_n = \left\lceil \frac{3}{8}(n-1)^2 \right\rceil + \left\lfloor \frac{1}{2}(n-1) \right\rfloor. \tag{3}$$

If n is odd,

$$\begin{aligned} M_n &= \left\lceil \frac{3}{8}\left((n^2-1) - 2(n-1)\right) \right\rceil + \frac{1}{2}(n-1) \\ &= \frac{3}{8}(n^2-1) - \left\lfloor \frac{3}{4}(n-1) \right\rfloor + \frac{1}{2}(n-1) \\ &= \frac{3}{8}(n^2-1) - \left\lfloor \frac{1}{4}(n-1) \right\rfloor, \end{aligned}$$

since $\lceil -x \rceil = -\lfloor x \rfloor$ for any $x \in \mathbb{R}$.

If n is even,

$$M_n = \left\lceil \frac{3}{8}\left(n(n-2) + 1\right) \right\rceil + \frac{1}{2}n - 1 = \frac{3}{8}n(n-2) + \frac{1}{2}n = \frac{1}{8}n(3n-2).$$

Thus

$$M_n = \begin{cases} \dfrac{3}{8}(n^2-1) - \left\lfloor \dfrac{1}{4}(n-1) \right\rfloor & \text{if } n \text{ is odd;} \\ \dfrac{1}{8}n(3n-2) & \text{if } n \text{ is even.} \end{cases}$$

When n is odd, $f(r, s)$ is maximized at $r = s = \frac{1}{2}(n-1)$, and equals

$$2\left\lfloor \frac{1}{16}(n-3)^2 \right\rfloor + \frac{1}{4}(n-1)^2. \tag{4}$$

If $n \equiv 1 \pmod 4$, $\lfloor \frac{1}{16}(n-3)^2 \rfloor = \frac{1}{16}(n-1)^2 - \lceil \frac{1}{4}(n-2) \rceil = \frac{1}{16}(n-1)^2 - \frac{1}{4}(n-1)$, and the expression in (4) equals $\frac{3}{8}(n-1)^2 - \frac{1}{2}(n-1) = \frac{3}{8}(n^2-1) - \frac{5}{4}(n-1)$.

If $n \equiv 3 \pmod 4$, $\lfloor \frac{1}{16}(n-3)^2 \rfloor = \frac{1}{16}(n-3)^2 = \frac{1}{16}(n-1)^2 - \frac{1}{4}(n-2)$, and the expression in (4) equals $\frac{3}{8}(n-1)^2 - \frac{1}{2}(n-1) + \frac{1}{2} = \frac{3}{8}(n^2-1) - \frac{5}{4}(n-1) + \frac{1}{2}$.

When n is even, $f(r, s)$ is maximized when $\{r, s\} = \{\frac{1}{2}n, \frac{1}{2}n - 1\}$, and equals

$$\left\lfloor \frac{1}{16}(n-2)^2 \right\rfloor + \left\lfloor \frac{1}{16}(n-4)^2 \right\rfloor + \frac{1}{4}n(n-2). \tag{5}$$

If $n \equiv 0 \pmod 4$, $\lfloor \frac{1}{16}(n-2)^2 \rfloor + \lfloor \frac{1}{16}(n-4)^2 \rfloor = \frac{1}{8}(n-4)^2 + \frac{1}{4}(n-4)$, and the expression in (5) equals $\frac{1}{8}n(3n-2) - (n-1)$.

If $n \equiv 2 \pmod 4$, $\lfloor \frac{1}{16}(n-2)^2 \rfloor + \lfloor \frac{1}{16}(n-4)^2 \rfloor = \frac{1}{8}(n-2)^2 - \frac{1}{4}(n-2)$, and the expression in (5) equals $\frac{1}{8}n(3n-2) - (n-1)$.

We now use (2) to show that the equation $f(r, s) = M_n - n$ has a solution in nonnegative integers r, s with $r + s = n - 1$ if and only if $n \not\equiv 1 \pmod 4$. We know that $f(r, s)$ attains its maximum value $M_n - (n-1)$ when r, s satisfy $r + s = n - 1$ and $|r - s| \le 1$. From (2), the next largest value attained by $f(r, s)$ for fixed $r + s$ is obtained by replacing $\max\{r, s\}$ by $\max\{r, s\} + 1$ and $\min\{r, s\}$ by $\min\{r, s\} - 1$.

Suppose that n is odd. With $r = s = \frac{1}{2}(n-1)$ in (2), we have

$$f\left(\frac{1}{2}(n-3), \frac{1}{2}(n+1)\right) = M_n - (n-1) - \left(\left\lceil \frac{1}{4}(n-1) \right\rceil - \left\lceil \frac{1}{4}(n-3) \right\rceil\right)$$
$$= \begin{cases} M_n - (n-1) & \text{if } n \equiv 1 \pmod 4; \\ M_n - n & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

If $n \equiv 1 \pmod 4$, from (2) with $r = \frac{1}{2}(n-3)$ and $s = \frac{1}{2}(n+1)$, we have

$$f\left(\frac{1}{2}(n-5), \frac{1}{2}(n+3)\right) = f\left(\frac{1}{2}(n-3), \frac{1}{2}(n+1)\right) + \left(\left\lceil \frac{1}{4}(n+1) \right\rceil - \left\lfloor \frac{1}{4}(n-5) \right\rfloor\right)$$
$$= M_n - (n+1).$$

Thus $f(r, s)$ cannot attain the value $M_n - n$ when $r + s = n - 1$ and $n \equiv 1 \pmod 4$.

Suppose that n is even. With $r = \frac{1}{2}n - 1$ and $s = \frac{1}{2}n$ in (2), we have

$$f\left(\frac{1}{2}n - 2, \frac{1}{2}n + 1\right) = M_n - (n-1) - \left(\left\lceil \frac{1}{4}n \right\rceil - \left\lceil \frac{1}{4}n - 1 \right\rceil\right) = M_n - n.$$

This completes the proof of our claim that $f(r, s) = M_n - n$ has a solution in nonnegative integers r, s with $r + s = n - 1$ if and only if $n \not\equiv 1 \pmod 4$.

We summarize our results for the function $f(r, s)$ as the following lemma.

Lemma 3.1. *For nonnegative integers r, s, define $f(r, s) = \lfloor \frac{1}{4}(r-1)^2 \rfloor + \lfloor \frac{1}{4}(s-1)^2 \rfloor + rs$. Let $M_n = \lceil \frac{3}{8}(n-1)^2 \rceil + \lfloor \frac{1}{2}(n-1) \rfloor$.*

(i) *If $r + s$ is the fixed positive integer $n - 1$, then $f(r, s)$ attains the maximum value $M_n - n + 1$. This maximum value is attained when $r = s = \frac{1}{2}(n-1)$ for odd n, also when $\{r, s\} = \{\frac{1}{2}(n-3), \frac{1}{2}(n+1)\}$ when $n \equiv 1 \pmod 4$, and when $\{r, s\} = \{\frac{1}{2}n - 1, \frac{1}{2}n\}$ for even n.*

(ii) *If $r + s = n - 1$, the equation $f(r, s) = M_n - n$ has a solution if and only if $n \not\equiv 1 \pmod 4$. The solution is given by $\{r, s\} = \{\frac{1}{2}(n-3), \frac{1}{2}(n+1)\}$ when $n \equiv 3 \pmod 4$, and by $\{r, s\} = \{\frac{1}{2}n - 2, \frac{1}{2}n + 1\}$ when n is even.*

Theorem 2.1 ensures the existence of an (n, m_n)-integral sum graph, since sum graphs are also integral sum graphs. In order to extend the range of size of integral sum graphs of order n, it is useful to note that

$$M_n - m_n = \left(\left\lceil \frac{3}{8}(n-1)^2 \right\rceil + \left\lfloor \frac{1}{2}(n-1) \right\rfloor\right) - \left\lfloor \frac{1}{4}(n-1)^2 \right\rfloor \ge \left\lfloor \frac{1}{8}(n-1)^2 \right\rfloor + \left\lfloor \frac{1}{2}(n-1) \right\rfloor$$

is increasing with n, and $M_3 - m_3 = 2$.

Lemma 3.2. *Any (n, m)-integral sum graph induced by a set of integers containing 0 can be extended to a $(n+2, m+2)$-integral sum graph and a $(n+2, m+3)$-integral sum graph.*

Proof. Let G be an (n, m)-integral sum graph, induced by the set of integers $\mathcal{L}(G)$ containing 0, with largest element in absolute value ℓ. Then the integral sum graph G' induced by the set $\mathcal{L}(G) \cup \{2\ell + 1, 3\ell + 2\}$ is an $(n+2, m+2)$-integral sum graph and the integral sum graph G'' induced by the set $\mathcal{L}(G) \cup \{-2\ell - 1, 2\ell + 1\}$ is an $(n+2, m+3)$-integral sum graph. In each case, the two additional vertices are adjacent to the vertex labelled 0, and non-adjacent vertices in G remain non-adjacent in G' and in G''. Moreover, the two additional vertices are not adjacent in G' but adjacent in G''. □

Theorem 3.1. *The maximum size for an integral sum graph of order n is $M_n = \lceil \frac{3}{8}(n-1)^2 \rceil + \lfloor \frac{1}{2}(n-1) \rfloor$. Moreover, there exists a sum graph of order n and size m, for each $m \le M_n$, except for $m = M_n - 1$ when $n \equiv 1 \pmod 4$.*

Proof. Throughout this proof we write $M_n = \lceil \frac{3}{8}(n-1)^2 \rceil + \lfloor \frac{1}{2}(n-1) \rfloor$ and $m_n = \lfloor \frac{1}{4}(n-1)^2 \rfloor$.

I. We first show that M_n is an upper bound for the size of an integral sum graph of order n. Let G_S be an integral sum graph of order n induced by a set of integers S of size n. Suppose S contains r positive and s negative integers, and $0 \notin S$. Let $x_1, \dots, x_r$ be vertices labelled with positive integers $a_1, \dots, a_r$ and vertices $y_1, \dots, y_s$ be labelled with negative integers $b_1, \dots, b_s$.

Then the maximum number of edges of the type x_iy_j is rs, those of the type x_ix_j is $m_r = \lfloor \frac{1}{4}(r-1)^2 \rfloor$, and those of the type y_iy_j is $m_s = \lfloor \frac{1}{4}(s-1)^2 \rfloor$. Thus the number of edges in G_S is no more than

$$\left\lfloor \frac{1}{4}(r-1)^2 \right\rfloor + \left\lfloor \frac{1}{4}(s-1)^2 \right\rfloor + rs = f(r,s), \tag{6}$$

with $r+s=n$. By Lemma 3.1, $f(r,s)$ attains a maximum $M_{n+1}-n$, since $r+s=n$. A simple calculation shows that

$$M_{n+1} - M_n = \begin{cases} \frac{1}{2}(n+1) + \left\lfloor \frac{1}{4}(n-1) \right\rfloor & \text{if } n \text{ is odd;} \\ n - \left\lfloor \frac{1}{4}n \right\rfloor & \text{if } n \text{ is even.} \end{cases}$$

Hence $M_{n+1} - n \le M_n$ for all $n \ge 1$. On the other hand, if $0 \in S$, G_S can have no more than $f(r,s) + n - 1$ edges with $r+s = n-1$, the additional $n-1$ edges being those with one endpoint corresponding to the vertex labelled 0. Lemma 3.1 shows this maximum to be M_n. Therefore the number of edges in G_S is *at most* M_n.

II. We next show the existence of an (n, M_n)-integral sum graph. We treat the two cases of odd and even n separately.

For odd n, consider a graph $\hat{\mathscr{G}}_n$ with vertices $u_1, \ldots, u_{(n-1)/2}, v_0, w_1, \ldots, w_{(n-1)/2}$, induced by the set of integers in $[-\frac{1}{2}(n-1), \frac{1}{2}(n-1)]$. Label the vertices by $\ell(u_i) = -i$ and $\ell(w_i) = i$ for $1 \le i \le \frac{1}{2}(n-1)$, and $\ell(v_0) = 0$. Thus $x \leftrightarrow y$ if and only if $-\frac{1}{2}(n-1) \le \ell(x) + \ell(y) \le \frac{1}{2}(n-1)$. Hence $u_i \leftrightarrow w_j$ for each pair $i, j \le \frac{1}{2}(n-1)$; there are $\frac{1}{4}(n-1)^2$ such edges. The graph induced by the vertices $w_1, \ldots, w_{(n-1)/2}$ is the sum graph $\mathscr{G}_{(n-1)/2}$ of maximum size of Theorem 2.1. Therefore the number of edges w_iw_j equals $\lfloor \frac{1}{16}(n-3)^2 \rfloor$, and there are an equal number of edges u_iu_j. In addition, there are $n-1$ edges with one endpoint v_0. Thus the number of edges in $\hat{\mathscr{G}}_n$ is $\frac{1}{4}(n-1)^2 + 2\lfloor \frac{1}{16}(n-3)^2 \rfloor + (n-1) = \max\{f(r,s) : r+s = n-1\} + (n-1) = M_n$, by (4).

For even n, consider a graph $\hat{\mathscr{H}}_n$ with vertices $u_1, \ldots, u_{(n-2)/2}, v_0, w_1, \ldots, w_{n/2}$, induced by the set of integers in $[-\frac{1}{2}n+1, \frac{1}{2}n]$. Label the vertices by $\ell(u_i) = -i$ for $1 \le i \le \frac{1}{2}n-1$, $\ell(w_i) = i$ for $1 \le i \le \frac{1}{2}n$, and $\ell(v_0) = 0$. Thus $x \leftrightarrow y$ if and only if $-\frac{1}{2}n+1 \le \ell(x)+\ell(y) \le \frac{1}{2}n$. Hence $u_i \leftrightarrow w_j$ for each pair i, j with $1 \le i \le \frac{1}{2}n-1$ and $1 \le j \le \frac{1}{2}n$; there are $\frac{1}{4}n(n-2)$ such edges. The graphs induced by the two sets of vertices $u_1, \ldots, u_{(n-2)/2}$ and $w_1, \ldots, w_{n/2}$ are the sum graphs $\mathscr{G}_{(n-2)/2}$ and $\mathscr{G}_{n/2}$ respectively, of maximum size defined in Theorem 2.1. Therefore the number of edges u_iu_j equals $\lfloor \frac{1}{16}(n-4)^2 \rfloor$ and the number of edges w_iw_j equals $\lfloor \frac{1}{16}(n-2)^2 \rfloor$. In addition, there are $n-1$ edges with one endpoint v_0. Thus the number of edges in $\hat{\mathscr{H}}_n$ is $\frac{1}{4}n(n-2) + \lfloor \frac{1}{16}(n-2)^2 \rfloor + \lfloor \frac{1}{16}(n-4)^2 \rfloor + (n-1) = \max\{f(r,s) : r+s = n-1\} + (n-1) = M_n$, by (5).

Thus $\hat{\mathscr{G}}_n$ is an (n, M_n)-integral sum graph for odd n while $\hat{\mathscr{H}}_n$ is an (n, M_n)-integral sum graph for even n.

III. We next show the existence of (n, m)-integral sum graphs for $n > 2$ and $0 \le m < M_n - 1$. Recall that the graphs $\hat{\mathscr{G}}_n$ and $\hat{\mathscr{H}}_n$ are integral sum graphs of order n and size M_n for odd and even n, respectively. We consider the two constructions separately.

Case 1. If n is odd, $n > 1$, $M_n - M_{n-2}$ equals $\frac{3}{2}(n-1) - 1$ if $n \equiv 1 (\text{mod } 4)$ and $\frac{3}{2}(n-1)$ if $n \equiv 3 (\text{mod } 4)$.

For n odd, $n > 1$, and $k \in [1, \frac{3}{2}(n-1) - 2]$, we construct a graph $G_{n,k}$ of order n and size $M_{n-2} + k$. As k varies over the set $\{1, \ldots, \frac{3}{2}(n-1) - 2\}$, this covers all integers in the interval $[M_{n-2}+1, M_n - 2]$, since $M_n - M_{n-2} \le \frac{3}{2}(n-1)$.

We use Lemma 3.2 and induction to construct graphs $G_{n,k}$ for $k \in \{1, 2, 3\}$. For $n = 3$, these are the graphs of sizes 1, 2 and 3, induced by the sets $\{1, 2, 3\}$, $\{-1, 0, 2\}$ and $\{-1, 0, 1\}$, respectively. Assume there exist $(k-2, M_{k-2} - 2)$-integral sum graphs for k, $3 \le k \le n+1$. In particular, assume the existence of a $(n-2, M_{n-2} - 2)$-integral sum graph. This can be extended to a $(n, M_{n-2}+1)$-integral sum graph, while the $(n-2, M_{n-2})$-integral sum graph (which exists by part II) can be extended to a $(n, M_{n-2}+2)$-integral sum graph and to a $(n, M_{n-2}+3)$-integral sum graph, by Lemma 3.2. Thus the claim holds for $k = 1, 2, 3$, and we assume that $k \ge 4$ for the rest of the proof of this part.

Consider a graph $G_{x,y}$ with vertices $u_1, \ldots, u_{(n-3)/2}, w_0, v_1, \ldots, v_{(n-3)/2}, u, v$, induced by the set of integers in $\left[-\frac{1}{2}(n-3), \frac{1}{2}(n-3)\right] \cup \{-x, y\}$, where $x, y \ge \frac{1}{2}(n-1)$ will be specified later. Label the vertices by $\ell(u_i) = -i$ for $1 \le i \le \frac{1}{2}(n-3)$, $\ell(v_j) = j$ for $1 \le j \le \frac{1}{2}(n-3)$, $\ell(w_0) = 0$, $\ell(u) = -x$ and $\ell(v) = y$.

The subgraph induced by vertices in the set of integers in $\left[-\frac{1}{2}(n-3), \frac{1}{2}(n-3)\right]$ is the graph $\hat{\mathscr{G}}_{n-2}$; this has size M_{n-2}. The remaining edges in $G_{x,y}$ are: (i) u_iu_j with $i+j = x$; (ii) v_iv_j with $i+j = y$; (iii) uv_j with $-\frac{1}{2}(n-3) \le j - x < 0$; (iv) vu_i with $0 < y - i \le \frac{1}{2}(n-3)$; (v) uv exactly when $|x-y| \le \frac{1}{2}(n-3)$; (vi) uw_0; and (vii) vw_0.

To count the number of edges in cases (i)–(iv), assume $i < j$. In case (i), $i+1 \le j = x - i \le \frac{1}{2}(n-3)$, so that $i \in \{x - \frac{1}{2}(n-3), \ldots, \lfloor \frac{1}{2}(x-1) \rfloor\}$. There are $\frac{1}{2}(n-3) - \lceil \frac{1}{2}(x-1) \rceil$ choices for i, and this is valid if $x \le n-2$. Similarly, in case (ii), there are $\frac{1}{2}(n-3) - \lceil \frac{1}{2}(y-1) \rceil$ choices for i, valid if $y \le n-2$. In case (iii), $j \in \{x - \frac{1}{2}(n-3), \ldots, \frac{1}{2}(n-3)\}$. There are $n-x-2$ choices for j, valid if $x \le n-2$. Similarly, in case (iv), there are $n-y-2$ choices for i, valid if $y \le n-2$. We shall choose x, y such that $|x-y| \le 2$ to ensure that $u \leftrightarrow v$.

In order that $G_{x,y}$ has $M_{n-2} + k$ edges, we need to show that there exist integers $x, y \in \left[\frac{1}{2}(n-1), n-2\right]$ such that

$$\left(x + \left\lceil \frac{1}{2}(x-1) \right\rceil\right) + \left(y + \left\lceil \frac{1}{2}(y-1) \right\rceil\right) = 3n - k - 4. \tag{7}$$

Since

$$f(x) = x + \left\lceil \frac{1}{2}(x-1) \right\rceil = \begin{cases} \frac{3}{2}(x-1)+1 & \text{if } x \text{ is odd;} \\ \frac{3}{2}x & \text{if } x \text{ is even,} \end{cases}$$

$f(2s) + f(2t) = 3(s+t)$, $f(2s+1) + f(2t) = 3(s+t)+1$, and $f(2s+1) + f(2t+1) = 3(s+t)+2$.

To see that $f(x)+f(y) = 3n-k-4 = 3(n-1)-(k+1)$ always has a solution with $x, y \in \left[\frac{1}{2}(n-1), n-2\right]$, write $k+1 = 3q - r$ with $r \in \{0, 1, 2\}$. Then $f(x) + f(y) = 3(n-1-q) + r$; the parity of the pair x, y is thus determined from $r \pmod 3$.

If $r = 0$, x and y must both be even; write $x = 2s$ and $y = 2t$. Then $s + t = n - 1 - q$ and $2 \le q \le \frac{1}{2}(n-3)$. Then $n + 1 \le x + y \le 2(n-3)$. Choose s, t such that $|s - t| \le 1$.

If $r = 1$, x and y must be of opposite parity; write $x = 2s$ and $y = 2t + 1$. Again $s + t = n - 1 - q$ but $2 \le q \le \frac{1}{2}(n-1)$. Hence $n \le x + y \le 2n - 5$. Choose x, y such that $|x - y| = 1$.

If $r = 2$, x and y must both be odd; write $x = 2s + 1$ and $y = 2t + 1$. Again $s + t = n - 1 - q$ but $3 \le q \le \frac{1}{2}(n-1)$. Hence $n + 1 \le x + y \le 2(n-3)$. Choose s, t such that $|s - t| \le 1$.

In all cases, there is a solution to (7) with $x, y \in \left[\frac{1}{2}(n-1), n-2\right]$. This completes the case when n is odd.

Case 2. If n is even, $n > 2$, then $M_n - M_{n-2} = \frac{3}{2}n - 2$.

For n even, $n > 2$, and $k \in [1, \frac{3}{2}n - 4]$, we construct a graph $H_{n,k}$ of order n and size $M_{n-2} + k$. As k varies over the set $\{1, \ldots, \frac{3}{2}n - 4\}$, this covers all integers in the interval $[M_{n-2} + 1, M_n - 2]$, since $M_n - M_{n-2} = \frac{3}{2}n - 2$.

As in Case 1, we use Lemma 3.2 and induction to construct graphs $H_{n,k}$ for $k \in \{1, 2, 3\}$. For $n = 4$, these are the graphs of sizes 2, 3 and 4, induced by the sets $\{1, 2, 3, 4\}$, $\{-1, 0, 2, 4\}$ and $\{-1, 0, 1, 3\}$, respectively. The rest of the argument is identical to the one in Case 1, and so we again assume that $k \ge 4$ for the rest of the proof of this part.

Consider a graph $H_{x,y}$ with vertices $u_1, \ldots, u_{(n-4)/2}, w_0, v_1, \ldots, v_{(n-2)/2}, u, v$, induced by the set of integers in $\left[-\frac{1}{2}(n-4), \frac{1}{2}(n-2)\right] \cup \{-x, y\}$, where $x \in \left[\frac{1}{2}(n-2), n-3\right]$ and $y \in \left[\frac{1}{2}n, n-3\right]$ will be specified later. Label the vertices by $\ell(u_i) = -i$ for $1 \le i \le \frac{1}{2}(n-4)$, $\ell(v_j) = j$ for $1 \le j \le \frac{1}{2}(n-2)$, $\ell(w_0) = 0$, $\ell(u) = -x$ and $\ell(v) = y$.

The subgraph induced by vertices in the set of integers in $\left[-\frac{1}{2}(n-4), \frac{1}{2}(n-2)\right]$ is the graph $\hat{\mathscr{H}}_{n-2}$; this has size M_{n-2}. The remaining edges in $H_{x,y}$ are: (i) u_iu_j with $i+j = x$; (ii) v_iv_j with $i+j = y$; (iii) uv_j with $-\frac{1}{2}(n-4) \le j-x < 0$; (iv) vu_i with $0 < y-i \le \frac{1}{2}(n-2)$; (v) uv exactly when either $x - y$ or $y - x$ belongs to $\left[-\frac{1}{2}(n-4), \frac{1}{2}(n-2)\right]$; (vi) uw_0; and (vii) vw_0.

To count the number of edges in cases (i)–(iv), assume $i < j$. In case (i), $i + 1 \le j = x - i \le \frac{1}{2}(n-4)$, so that $i \in \{x - \frac{1}{2}(n-4), \ldots, \lfloor \frac{1}{2}(x-1) \rfloor\}$. There are $\frac{1}{2}(n-4) - \lceil \frac{1}{2}(x-1) \rceil$ such edges, and this is valid if $x \le n-3$. In case (ii), $i \in \{y - \frac{1}{2}(n-2), \ldots, \lfloor \frac{1}{2}(y-1) \rfloor\}$. There are $\frac{1}{2}(n-2) - \lceil \frac{1}{2}(y-1) \rceil$ such edges, and this is valid if $y \le n-1$. In case (iii), $j \in \{x - \frac{1}{2}(n-4), \ldots, \frac{1}{2}(n-2)\}$; there are $n-x-2$ such edges, valid if $x \le n-2$. In case (iv), $i \in \{y - \frac{1}{2}(n-2), \ldots, \frac{1}{2}(n-4)\}$; there are $n-y-2$ such edges, valid if $y \le n-2$. As in Case 1, we shall choose x, y such that $|x-y| \le 2$ to ensure that $u \leftrightarrow v$.

In order that $H_{x,y}$ has $M_{n-2} + k$ edges, we need to show that there exist integers $x \in \left[\frac{1}{2}n - 1, n-3\right]$ and $y \in \left[\frac{1}{2}n, n-2\right]$ such that

$$\left(x + \left\lceil \frac{1}{2}(x-1) \right\rceil\right) + \left(y + \left\lceil \frac{1}{2}(y-1) \right\rceil\right) = 3n - k - 4.$$

This is Eq. (7) of Case 1, and we follow the same argument as above. If $r = 0$, x and y must both be even; write $x = 2s$ and $y = 2t$. Then $s + t = n - 1 - q$ and $2 \le q \le \frac{1}{2}n - 1$. Then $n \le x + y \le 2(n-3)$. Choose s, t such that $s = t$ or $t - 1$.

If $r = 1$, x and y must be of opposite parity; write $x = 2s$ and $y = 2t + 1$. Again $s + t = n - 1 - q$ but $2 \le q \le \frac{1}{2}n$. Hence $n - 1 \le x + y \le 2n - 5$. Choose $s = t$.

If $r = 2$, x and y must both be odd; write $x = 2s + 1$ and $y = 2t + 1$. Again $s + t = n - 1 - q$ but $3 \le q \le \frac{1}{2}n$. Hence $n \le x + y \le 2(n-3)$. Choose s, t such that $s = t$ or $s = t - 1$.

In all cases, there is a solution to (7) with $x \in \left[\frac{1}{2}n - 1, n-3\right]$ and $y \in \left[\frac{1}{2}n, n-2\right]$. This completes the case when n is even.

We have so far shown that there exists an (n, m)-integral sum graph whenever $m \in [M_{n-2}, M_n - 2]$. By Theorem 2.1, there exist (n, m)-sum graphs (hence, integral sum graphs) for each nonnegative integer $m \le m_n$. To extend the range of m to include all nonnegative integers less than M_{n-2}, we use induction on n. Since a sum graph is also an integral sum graph, and $m_3 = M_3 - 2$, this verifies the case $n = 3$. Since $m_4 = M_4 - 3 = 2$, we need to construct an $(4, 3)$-integral sum graph to verify the case $n = 4$. The graph induced by the set $S = \{-1, 0, 2, 4\}$ works, thus verifying the claim for $n = 3$ and $n = 4$. Assume that there exists integral sum graphs of all orders k less than n, and for each k, of all sizes less than $M_k - 1$. The existence of an (n, m)-integral sum graph for $2 < m < M_{n-2}$ follows from the existence of an $(n-2, m-2)$-integral sum graph by Lemma 3.2 and induction hypothesis. Existence of integral sum graphs of order n and sizes 0 and 1 follow from the existence of such sum graphs. This completes the argument for the existence of (n, m)-integral sum graphs for $m < M_n - 1$.

IV. To complete the proof, we need to show the existence of an $(n, M_n - 1)$-integral sum graph precisely for $n \not\equiv 1 \pmod 4$. Lemma 3.1 provides examples of such graphs when $n \not\equiv 1 \pmod 4$, and also shows that none exist if the graph is induced by a set of integers that contain 0 when $n \equiv 1 \pmod 4$.

Suppose $n \equiv 1 (\mathrm{mod}\ 4)$ and G_S is an integral sum graph induced by a set S consisting of r positive and s negative integers, with $r + s = n$. If either r or s is 0, the maximum size of G_S is m_n and this is less than $M_n - 1$. So we may assume that there exist at least one vertex with a positive label and one with a negative label. Since the vertex with the smallest positive label cannot be adjacent to the one with the largest negative label (since $0 \notin S$), the maximum possible number of edges in G_S is $m_r + m_s + rs - 1 = f(r, s) - 1$. This is *at most* $M_{n+1} - n - 1$. Recall that

$$M_{n+1} - M_n = \frac{1}{2}(n+1) + \left\lfloor \frac{1}{4}(n-1) \right\rfloor$$

when n is odd. Since $M_{n+1} - M_n < n$ holds for all odd $n > 1$, it follows that $M_{n+1} - n - 1 < M_n - 1$ also holds for all odd $n > 1$. □

Remark 3.1. The maximum size in Theorem 3.1 reduces to $\frac{3}{8}(n^2 - 1) - \lfloor \frac{1}{4}(n-1) \rfloor$ when n is odd and to $\frac{1}{8}n(3n-2)$ when n is even.

We close this section by determining those sets of integers S which induce an (n, M_n)-integral sum graph. The constructions in Theorem 3.1 establish that $S = \left[-\frac{1}{2}(n-1), \frac{1}{2}(n-1)\right]$ for odd n and $S = \left[-\frac{1}{2}n + 1, \frac{1}{2}n\right]$ for even n is one such set, for each $n \geq 1$. Since the graphs induced by the sets S and cS are isomorphic for any $c \geq 1$ and any set S of integers, it suffices to consider only reduced sets S.

Theorem 3.2. *Let S be a reduced set of integers such that $|S| = n$ and $|E(G_S)| = M_n = \lceil \frac{3}{8}(n-1)^2 \rceil + \lfloor \frac{1}{2}(n-1) \rfloor$. Then the set S is given by*

$$\begin{cases} \{1\} & \text{if } n = 1; \\ \{0, 1\} & \text{if } n = 2; \\ \left[-\frac{1}{2}(n-1), \frac{1}{2}(n-1)\right], \left[-\frac{1}{2}(n+1), \frac{1}{2}(n-3)\right], \quad \text{or} \quad \left[-\frac{1}{2}(n-3), \frac{1}{2}(n+1)\right] & \text{if } n \text{ is odd}, n > 1; \\ \left[-\frac{1}{2}n + 1, \frac{1}{2}n\right] \text{ or } \left[-\frac{1}{2}n, \frac{1}{2}n - 1\right] & \text{if } n \text{ is even}, n > 2. \end{cases}$$

Proof. Throughout this proof we write $m_n = \lfloor \frac{1}{4}(n-1)^2 \rfloor$ and $M_n = \lceil \frac{3}{8}(n-1)^2 \rceil + \lfloor \frac{1}{2}(n-1) \rfloor$. The case where $n = 1$ is trivial, and the case where $n = 2$ follows from the observation that S must contain 0. In what follows, we assume $n \geq 3$.

We use S^+ (respectively, S^-) to denote the set of positive (respectively, negative) integers in S. If $0 \notin S$, we follow the argument in Theorem 3.1, part I, to observe that in order that G_S has M_n edges, we must have $\frac{1}{2}(n-1) = \lfloor \frac{1}{4}(n-1) \rfloor$ when n is odd and $\lfloor \frac{1}{4}n \rfloor = 0$ when n is even. This is impossible, since $n > 2$. So we may assume $0 \in S$, and again follow the argument in Theorem 3.1, part I. In order that G_S has M_n edges, the graphs induced by S^+ must have size m_r and that induced by S^- size m_s. In addition, each vertex with label in S^+ must be adjacent to each vertex with label in S^-—we call this the "cross-edges" condition. It is easy to see that the edges in the graph induced by an interval $[-p, q]$ containing 0 satisfies the cross-edge condition. On the other hand, the edges in the graph induced by $[-p, q] \setminus \{a\}$ $(a \neq -p, q)$ does not satisfy the cross-edge condition, since a can be realized as a sum of a positive and a negative integer in $[-p, q]$. Hence, if either S^+ or S^- is of the form $\{1, \ldots, n-1, n+1\}$, S cannot induce a graph which satisfies the cross-edge condition. So, for $n \geq 5$, $S^+ = \{1, \ldots, r\}$ and $-S^- = \{1, \ldots, s\}$ with $r + s = n - 1$. We now use Lemma 3.1 and Theorem 2.2.

Case I. If n is odd, then $r = s = \frac{1}{2}(n-1)$, and additionally $\{r, s\} = \{\frac{1}{2}(n-3), \frac{1}{2}(n+1)\}$ if $n \equiv 1 \pmod 4$, by Lemma 3.1.

By Theorem 2.2, $S^+ = \{1, \ldots, \frac{1}{2}(n-1)\} = -S^-$. This gives $S = [-\frac{1}{2}(n-1), \frac{1}{2}(n-1)]$, which induces an integral sum graph of size M_n. The additional case applies only when $\frac{1}{2}(n-1)$ is even, and in this case, either $S^+ = \{1, \ldots, \frac{1}{2}(n-3)\}$, $-S^- = \{1, \ldots, \frac{1}{2}(n+1)\}$ or $S^+ = \{1, \ldots, \frac{1}{2}(n+3)\}$, $-S^- = \{1, \ldots, \frac{1}{2}(n-3)\}$. These result in $S = [-\frac{1}{2}(n+1), \frac{1}{2}(n-3)]$ and $S = [-\frac{1}{2}(n-3), \frac{1}{2}(n+1)]$.

Case II. If n is even, then $\{r, s\} = \{\frac{1}{2}n - 1, \frac{1}{2}n\}$, by Lemma 3.1.

By Theorem 2.2, either $S^+ = \{1, \ldots, \frac{1}{2}n - 1\}$, $-S^- = \{1, \ldots, \frac{1}{2}n\}$ or $S^+ = \{1, \ldots, \frac{1}{2}n\}$, $-S^- = \{1, \ldots, \frac{1}{2}n - 1\}$. These result in $S = [-\frac{1}{2}n + 1, \frac{1}{2}n]$ and $S = [-\frac{1}{2}n, \frac{1}{2}n - 1]$.

This completes the characterization of sets of integers of size n that induce an integral sum graph of size M_n. □

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