The appendix to “Does stronger protection of intellectual property stimulate innovation?”

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In this paper, we derive the various expressions posited in Gangopadhyay and Mondal (2012).

A Derivation of Equation (7)

We shall derive equation (7). The no-arbitrage condition in equation (6) can be written as $q \frac{\pi _M}{v} = \rho + qg$. In this formulation, we make use of the fact that $\frac{v}{v} = -g$ and $r = \rho$. Using the expressions of profit, $\pi _M = \frac{1-\alpha}{\alpha} wx_M$, and value of a firm, $v = \frac{a}{n_M + \lambda n_C}$, in equation (6), we obtain:

$$q \frac{1 - \alpha}{a\alpha} x_M (n_M + \lambda n_C) = \rho + qg.$$  

Next, we replace $n_C$ in terms of $n_M$ as $n_C = \frac{1-q}{q} n_M$, we have:

$$q \frac{1 - \alpha}{a\alpha} n_M x_M \left( 1 + \frac{1-q}{q} \lambda \right) = \rho + qg.$$  

Rearranging the terms of this equation, we obtain equation (7).

B Proof of Proposition 1

We differentiate equation (9) with respect to $q$ to get

$$\frac{dq}{dq} = \frac{dN}{dq} D - \frac{dD}{dq} N \frac{D}{D^2}.$$  

Clearly $\frac{dD}{dq} > 0$ since $\alpha^{-\epsilon} > 1$. We, now, find out the condition for which $\frac{dN}{dq} \leq 0$ and $N > 0$. Differentiating the expression of $N$ with respect to $q$, we get

$$\frac{dN}{dq} = \frac{\rho}{q^2} \frac{aa\alpha}{1-\alpha} - \frac{\lambda - \alpha^{-\epsilon}}{(q + \alpha^{-\epsilon}(1-q))^2}.$$  

We assume that $\lambda$ is sufficiently large so that

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\[ \lambda > \alpha^{-\varepsilon}. \] \hfill (A-1)

Then \( \frac{dN}{dq} < 0 \) if
\[ q > \frac{\alpha^{-\varepsilon}}{\alpha^{-\varepsilon} - 1 + \sqrt{(1-\alpha)L/(a\rho \alpha)} (\lambda - \alpha^{-\varepsilon})} \quad (\equiv q^*). \] \hfill (A-2)

In inequality (A-2), the right hand side has to be less than unity as \( q \) can not be greater than unity. This puts a restriction on \( q^* \), namely \( q^* < 1 \), i.e.,
\[ (1-\alpha)L/a > \frac{\alpha \rho}{\lambda - \alpha^{-\varepsilon}}. \] \hfill (A-3)

Finally, it is easy to verify that \( \frac{d^2N}{dq^2} < 0 \) for all values of \( q \) as long as \( \lambda > \alpha^{-\varepsilon} \) is satisfied. Then \( N \) has a global maximum at \( q = q^* \). The argument for this goes as follows: \( N \) approaches to negative infinity when \( q \to 0 \). Also at \( q = 1 \), \( N \) takes the value \( N = L - \frac{a\rho}{1-\alpha} \). This value of \( N \) is positive if \((1-\alpha)L/a > \rho \alpha \). This inequality and the inequality (A-3) are both satisfied, if the following is satisfied:
\[ (1-\alpha)L/a > \max\{\frac{\alpha \rho}{\lambda - \alpha^{-\varepsilon}}; \alpha \rho\}. \] \hfill (A-4)

(A-4) is a sufficient condition for positivity of \( N \) at \( q = q^* \). (A-4) and (A-1), when satisfied jointly, yields \( \frac{dN}{dq} < 0 \) for \( q \in (q^*, 1) \). Since \( N \) attains its maximum at \( q = q^* \) and \( N \) is positive at \( q = 1 \), we must have \( N > 0 \) for all values of \( q \in [q^*, 1] \). This ensures that \( g > 0 \) for \( q \in [q^*, 1] \).

C Inverted ‘U’ Shape

The expression for \( \frac{dg}{dq} \) approaches positive infinity for \( q \to 0 \) as \( \lim_{q \to 0} \frac{dN}{dq} = +\infty \) and \( \frac{dD}{dq} \) is positive and finite for all \( q \). Also \( \frac{dg}{dq} < 0 \) at \( q = q^* \) as \( \frac{dN}{dq} = 0 \) at that point. Then, by intermediate value theorem, there must exist a \( 0 < \bar{q} < q^* \) such that \( \frac{dg}{dq} = 0 \) at \( q = \bar{q} \). Also it is easy to check that
\[ \frac{d^2g}{dq^2} = \frac{(\frac{d^2N}{dq^2}) D - \left(\frac{dD}{dq}\right) N}{D^2} < 0; \forall q, \]

since \( \frac{d^2N}{dq^2} < 0 \) and \( \frac{dD}{dq^2} > 0 \) for all values of \( q \in (0, 1) \). Note that \( \frac{d^2N}{dq^2} < 0 \) requires the assumption that \( \lambda > \alpha^{-\varepsilon} \). We assume that this is always fulfilled. Also we are only focusing on the values of \( g \) such that \( g > 0 \). Then \( N \) must be positive which requires that \( q \) must not be too close to zero. This guarantees an inverse ‘U’ shape relationship between \( g \) and \( q \) when \( g > 0 \). Thus, \( g \) must attain the maximum at \( q = \bar{q} \) and this maximum value must be positive. The Latter is guaranteed from the fact that \( g \) is positive at \( q = q^* \). Therefore, the maximum value of \( g \) must be positive too.

D Model without Scale Effect

We introduce population growth, \( \gamma \), in this model. Then population at time \( t \), \( L(t) \) is given by \( L_0 \cdot e^{-\gamma t} \). Discounted lifetime utility of a representative individual in a household can be written as \( W = \int_{\tau}^{\infty} e^{-(\rho-\gamma)(\tau-t)} \log(U(\tau)) d\tau \), where the static utility has the same functional form as in section 2 in Gangopadhyay and Mondal (2012). The intertemporal budget constraint is \( A(t) = w(t) + r(t)A(t) - \gamma A(t) - e(t) \). Here, \( e(t) \) denotes the instantaneous expenditure of the
representative consumer. We need to assume that $\rho > \gamma$. Consumer’s optimization exercise gives the standard Euler’s equation $\frac{\psi_t}{n(t)} = r(t) - \rho$; and normalizing the expenditure to unity, we obtain $r(t) = \rho$. We use the knowledge spillover term in the R&D sector as 

$$K(n_M, n_C) = (n_M + \lambda n_C)^\phi$$  \hspace{1cm} (A-5)$$

where $0 < \phi < 1$. As in the model in the text, the instantaneous profit of a monopoly firm is given by $\pi_M$. But equation (5) is now modified to be $C = \frac{a}{(n_M + \lambda n_C)^\phi}w$. Since free entry condition with ongoing R&D implies that $C = v$, we must have $\frac{v}{v} = -\phi g$ in the steady state. With this, equation (6) in the text can be in reformulated as $q\frac{\pi_M}{C} = \rho + \phi gg$. Using the expressions of $\pi_M$ and $v$, this can be written as $q \frac{1-\rho}{(n_M + \lambda n_C)^\phi}w = \rho + \phi gg$; and with some reformulation, we arrive at the following form:

$$n_M x_M = \frac{a \alpha}{1 - \alpha} (\rho + \phi gg) \frac{n^{1-\phi}}{(q + \lambda(1-q))^\phi}.$$  \hspace{1cm} (A-6)$$

From the labour market equilibrium condition, we get the following expression of ongoing R&D implies that $C$ by $\pi$ where $0 < \phi < 1$.

$$n_M x_M = \frac{L - L_R}{1 + \frac{1-q}{q} \alpha^{-\epsilon}} = \frac{L - a n g^{1-\phi}(q + \lambda(1-q))^{-\phi}}{1 + \frac{1-q}{q} \alpha^{-\epsilon}}.$$  \hspace{1cm} (A-7)$$

Using equations (A-6) and (A-7) and rearranging further one gets the following relationship:

$$\frac{L}{a n^{1-\phi}} = \frac{\alpha}{1 - \alpha} (\rho + \phi gg) \frac{1 + \frac{1-q}{q} \alpha^{-\epsilon}}{(q + \lambda(1-q))^\phi} + \frac{g}{(q + \lambda(1-q))^\phi}.$$  \hspace{1cm} (A-8)$$

We must have $g = \frac{2}{\phi}$ to ensure that the sectoral allocation of labour remain constant along the balanced growth path. This uniquely solves for the rate of innovation. Then, we plug in the value of $g$ in equation (A-8) to have an expression for $\frac{L}{n^{1-\phi}}$. We define a new variable $\delta$ such that $\delta = \frac{L}{n^{1-\phi}}$. One interpretation of $\delta$ is that it is an inverse measure of the ‘R&D difficulty index’. In the steady state, $\delta$ is constant. Equation (A-8) can now be written as

$$a \delta = \frac{\alpha}{1 - \alpha} (\rho + \phi gg) \frac{1 + \frac{1-q}{q} \alpha^{-\epsilon}}{(q + \lambda(1-q))^\phi} + \frac{g}{(q + \lambda(1-q))^\phi}.$$  \hspace{1cm} (A-9)$$

From equation (A-9), if an increase in $q$ increases $\delta$ then the rate of innovation has to decline temporarily. To show that this possibility can arise, define a new variable $Z(q)$ such that $Z(q) = \frac{1 + \frac{1-q}{q} \alpha^{-\epsilon}}{(q + \lambda(1-q))^\phi}$. Then, to have $\frac{dZ(q)}{dq} \geq 0$ is a sufficient condition for $a \delta = \frac{L}{n^{1-\phi}}$. The latter holds true if $q$ is sufficiently large and some other regularity conditions are satisfied. The proof goes as follows:

**Proof:**
To prove that $Z'(.) > 0$ (or, $Z'(.) \geq 0$) we differentiate log ($Z(q)$) w.r.t. $q$ to get

$$\frac{Z'(.)}{Z(.)} = \frac{(\lambda-1)\phi}{\alpha^{-\epsilon}} - \frac{1}{q} \frac{q + \lambda(1-q)}{q + \alpha^{-\epsilon}(1-q)}.$$ So $Z'(.) \geq 0$ if $\frac{1}{q} \frac{q + \lambda(1-q)}{q + \alpha^{-\epsilon}(1-q)} \leq \frac{(\lambda-1)\phi}{\alpha^{-\epsilon}}$. But the left hand side of this last inequality is a decreasing function of $q$ if $\lambda \geq \alpha^{-\epsilon}$. This curve is asymptotic to the vertical axis when $q$ approaches zero and reaches to the value unity at $q = 1$. The right hand side of this inequality is a constant with value greater than unity if $\lambda > 1 + \frac{\alpha^{-\epsilon}}{\phi}$. Then there exists a unique solution to the equation $\frac{1}{q} \frac{q + \lambda(1-q)}{q + \alpha^{-\epsilon}(1-q)} = \frac{(\lambda-1)\phi}{\alpha^{-\epsilon}}$ at $q = q^*$. For all $q \geq q^*$, we must have $Z'(.) \geq 0$.

So, under the following two conditions:

(i) $q \geq q^*$ (where $q^*$ solves the equation $\frac{1}{q} \frac{q + \lambda(1-q)}{q + \alpha^{-\epsilon}(1-q)} = \frac{(\lambda-1)\phi}{\alpha^{-\epsilon}}$)
(ii) $\lambda > 1 + \frac{\alpha^{-\epsilon}}{\phi}$,

we get the desired result that an increase in the strength of IPR protection may temporarily decrease the rate of innovation and permanently increase $\delta$.

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1We drop the time subscript from now on.
References