ELL800 Assignment (Non-graded)

January 12, 2017

1 Questions on Field

Question 0

List all the field axioms and try to find redundant axioms.

Question 1

Check if the set of integers $\mathbb{Z} := \{..., -2, -1, 0, 1, 2, 3, ...\}$ with usual addition and multiplication operation is a field by verifying field axioms. Which field axiom is not satisfied and why?

Question 2

Check if the set of real numbers $\mathbb{R}$, the set of complex numbers $\mathbb{C}$ and the set of rational numbers $\mathbb{Q}$ with usual addition and multiplication operation is a field by verifying the field axioms. Which field axiom is not satisfied and why?

Question 3

Let us consider a set of 2-tuples of real numbers i.e,

$$\mathbb{R}^2 := \{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}.$$

Define addition and multiplication operations in such a way that this set forms a field. Verify all the field axioms for this set to prove your claim.

Question 3

Let the remainder r be obtained by dividing an integer n by p. We will write it as $r \equiv n \bmod p$ which is read as, r is equivalent to n modulo integer p. For example $1 \equiv 4 \bmod 3$; because on dividing 4 by 3 we get 1 as a remainder. We can define addition of two numbers n_1 and n_2, modulo p, as follows

$$n_1 + n_2 := (n_1 + n_1) \bmod p.$$

Similarly, multiplication can be defined as

$$n_1 \cdot n_2 := (n_1 \cdot n_2) \mod p.$$

The set of remainders obtained by dividing any integer by another integer p is denoted by $\frac{\mathbb{Z}}{p\mathbb{Z}}$. For example, if $p = 2$, we get 0 and 1 as remainders and hence set $\frac{\mathbb{Z}}{2\mathbb{Z}}$ consists of two integers $\{0, 1\}$. Also, for $p = 4$, we get $0, 1, 2, 3$ as remainders and hence set $\frac{\mathbb{Z}}{4\mathbb{Z}}$ consists of four integers $\{0, 1, 2, 3\}$.Prove that under the addition and multiplication defined above the the set $\frac{\mathbb{Z}}{p\mathbb{Z}}$ is a field for $p = 2, 3, 5$. Is it a field for $p = 4, 6, 8, 10$? Verify the field axioms and check if these sets are field.

2 Vector Spaces

Question 0

List all the vector space axioms.

Question 1

Show that the n-tuple of real numbers defined as follows

$$\mathbb{R}^n := \{(x_1, x_2, ..., x_n) : x_i \in \mathbb{R} \text{ for } i = 1, 2, ..., n\}.$$

with component wise addition is a vector space by verifying the axioms of vector space.

Question 2

Show that set of all functions $f : \mathbb{R} \to \mathbb{R}$ with pointwise addition is a vector space.

Question 3

Show that set of all polynomials with usual polynomial addition is a vector space.

Question 4

Show that the set of all solutions to a linear constant coefficient homogenous differential equation

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + ... + a_0 x = 0$$

forms a vector space.

3 Span, Linear independence, Basis, Subspace

Question 0

Consider vectors $x_1 = (1, 1) \in \mathbb{R}^2$ and $x_2 = (2, 2) \in \mathbb{R}^2$. What is the span of these two vectors? Draw it. Are these two vectors linearly independent? Justify your answer.

Question 1

Express vector $u = (1, -1) \in \mathbb{R}^2$ in the span of $x_1 = (2, 2) \in \mathbb{R}^2$ and $x_2 = (4, 2) \in \mathbb{R}^2$. What is the span of x_1 and x_2? Draw it. Are these vectors linearly independent? Justify your answer.

Question 2

Express vector $u = (1, 1, -1) \in \mathbb{R}^3$ in the span of $x_1 = (2, 2, 3) \in \mathbb{R}^3$, $x_2 = (3, 3, 4) \in \mathbb{R}^3$ and $x_3 = (3, -3, 4) \in \mathbb{R}^3$. Are these vectors linearly independent? Justify your answer.

Question 3

Does the set $B = \{x_1, x_2, x_3\}$ where $x_1 = (1, 2, 3) \in \mathbb{R}^3$, $x_2 = (3, 3, 4) \in \mathbb{R}^3$ and $x_3 = (3, -3, 4) \in \mathbb{R}^3$ form a basis for $\mathbb{R}^3$? Why?

Question 4

Does the set $B = \{x_1, x_2, x_3\}$ where $x_1 = (2, 2, 3) \in \mathbb{R}^3$, $x_2 = (3, 3, 4) \in \mathbb{R}^3$ and $x_3 = (3, -3, 4) \in \mathbb{R}^3$ form a basis for $\mathbb{R}^3$? Why?

Question 5

Let $x_1 = (1, 2, 3) \in \mathbb{R}^3$ and $x_2 = (1, 3, 4) \in \mathbb{R}^3$. Show that the span $\{x_1, x_2\}$ is a subspace of $\mathbb{R}^3$. What is the dimension of the span of $\{x_1, x_2\}$?

Question 6

Show that all bases of a subspace S of $\mathbb{R}^n$ have the same number of elements which is equal to the dimension of the subspace.

Question 7

Let $S_1 = \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$. Determine dimension of S_1

Question 8

Let $S_1 = \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ and $S_2 = \text{span}\left\{\begin{bmatrix}0\\9\\0\end{bmatrix}, \begin{bmatrix}1\\4\\1\end{bmatrix}, \begin{bmatrix}2\\6\\2\end{bmatrix}\right\}$. Compute a basis for $S_1 + S_2$ and $S_1 \cap S_2$. Verify the identity $\dim(S_1 + S_2) = \dim S_1 + \dim S_2 - \dim(S_1 \cap S_2)$.

Question 9

Let

$$P_3 := \{p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \text{ such that } a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

be a set of all polynomials with degree less than or equal to 3. Let $\frac{d}{dx}$ be a derivative map which acts on a polynomial in P_3 and gives out a polynomial in P_3 i.e.,

$$\frac{d}{dx} : P_3 \to P_3.$$

Prove that this map is a linear transformation. Find a matrix representation for it. Find the basis for the image and kernel of this matrix.

Question 10

Given a matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & 8 & 13 \\ 5 & 8 & 13 & 21 & 34 \end{bmatrix}.$$

Find a basis for the image and the kernel of this matrix by using column operations. Determine the rank of this matrix from your calculations. Repeat the calculations for $A^\top$