Krylov Subspaces.

Given A ; $x \in \mathbb{R}^n$.

$$\mathcal{K}_1(A, x) = \mathrm{sp}\{x\}$$

$$\mathcal{K}_2(A, x) = \mathrm{sp}\{x, Ax\}$$

$$\vdots$$

$$\mathcal{K}_n(A, x) = \mathrm{sp}\{x, Ax, \dots, A^{n-1}x\}$$

The Arnoldi Process.

- Modification to Power method and Simultaneous iteration

$\mathcal{K}_{(K+1)}(A, q) = \mathrm{span}\{q, Aq, \dots, A^K q\}$ → usually illconditioned.

Suppose $\{q, Aq, \dots, A^{K-1}q\}$ orthogonalize
$q_1, q_2, \dots, q_K$
to obtain next vector we just multiply

$$A(A^{K-1})q = A^K q.$$

But we have now replaced by $q_1, \dots, q_K$

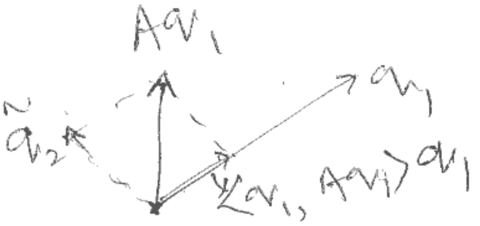
Step 1 $\quad q$

$$q_1 = \frac{q}{\|q\|_2}$$

step 2 $\quad Aq_1$

orthonormalize w.r.t q_1

$$\tilde{q}_2 = Aq_1 - \underbrace{\langle q_1, Aq_1 \rangle}_{h_{11}} q_1$$



$$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\| \rightarrow h_{21}}$$

$$Aq_1 = h_{21} q_2 + h_{11} q_1$$

Step 3. $\quad Aq_2$

orthonormalize w.r.t q_1, q_2

$$\tilde{q}_3 = Aq_2 - \underbrace{\langle q_1, Aq_2 \rangle}_{h_{12}} q_1 - \underbrace{\langle q_2, Aq_2 \rangle}_{h_{22}} q_2$$

$$q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|_2 \rightarrow h_{32}}$$

$$Aq_2 = h_{32} q_3 + h_{22} q_2 + h_{12} q_1$$

So on.

$$Aq_K = h_{K+1,K}\, q_{K+1} + h_{K,K}\, q_K + \dots + h_{1,K}\, q_1$$

$$A[q_1, q_2, \dots q_k] = [q_1 \quad \dots \quad q_{k+1}] \begin{bmatrix} h_{11} & h_{12} & \dots & h_1 \\ h_{21} & h_{22} & & h_2 \\ 0 & h_{32} & & h_3 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & h_{k+1} \end{bmatrix}$$

$$A Q_k = Q_{k+1} H_{k+1,k}.$$

$$\underline{A Q_k = Q_k H_k + q_{k+1} h_{k+1,k} e_k^T} \quad \text{— (1)}$$

Now if $h_{k+1,k} = 0$.

We have found an invariant subspace of A

and $\lambda(H_k) \subseteq \lambda(A)$

Thm: Let Q_k, H_k and $h_{k+1,k}$ be generated by Arnoldi Process. s.t. (1) holds.

Let $\mu \in \lambda(H_k)$ with x eigenvector. $\|x\|_2 =$

$v = Q_m x \in \mathbb{R}^n$ Ritz value, (μ, v) - Ritz Galerkin approx. of Eig.v. of A

$$\|Av - \mu v\|_2 = |h_{k+1,k}| \, |x_k|$$

x_k k^{th} Component of x

(μ, v) is Ritz pair of A. (Ritz Galerkin approx.

Arnoldi Process.

generates Krylov subspace orthogonal basis

$$A Q_m = Q_{m} H_m + q_{m+1} h_{m+1,m} e_m^T$$

$$A[q_1, q_2, \ldots, q_m] = [q_1, q_2, \ldots, q_m] H_m + q_{m+1} [0\ 0 \ldots 0\ h_{m+1,m}]$$

$$\left.\begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1,m-1} & h_{1m} \\ h_{21} & h_{22} & \cdots & h_{2,m-1} & h_{2m} \\ 0 & h_{32} & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & h_{m,m-1} & h_{m,m} \\ \hline 0 & 0 & \cdots & 0 & h_{m+1,m} \end{bmatrix}\right\} H_m.$$

$h_{m+1,m} = 0$ if and only if

$q_1, Aq_1, \ldots, A^{m-1}q_1, A^m q_1$, forms a dependent set of vectors

i.e. $A^m q_1 = -\alpha_0 q_1 - \alpha_1 A q_1 \cdots - \alpha_{m-1} A^{m-1} q_1$

and $(A^m + \alpha_{m-1} A^{m-1} + \cdots + \alpha_1 A + \alpha_0 I) q_1 = 0$

$p(s) = s^m + \alpha_{m-1} s^{m-1} + \cdots + \alpha_1 s + \alpha_0$ is minimal poly. of A w.r.t q_1

Thm: 6.4.16.

Let $\mu \in \lambda(H_m)$

with x as eigenvector with $\|x\|_2 = 1$

Let $v = Q_m x$

Then

$$A Q_m x = Q_m H_m x + q_{m+1} h_{m+1,m} e_m^T x$$

$$\Rightarrow A v = \mu v + q_{m+1} [h_{m+1,m} x_m]$$

$$\Rightarrow \|Av - \mu v\|_2 = |h_{m+1,m}| |x_m|$$

Thus $\|Av - \mu v\|_2 = 0$ if $h_{m+1,m} = 0$ or $x_m = 0$ or both.

in which case (μ, v) also forms an eigenpair of A

$v \rightarrow$ Ritz Vector of A associated with $K_m(A, q) = \text{Span}\{q_1, \ldots, q_m\}$
$= \text{Span}\{q_1, Aq_1, \ldots, A^{m-1}q_1\}$

$\downarrow$

Rayleigh - Ritz - Galerkin approx. to an eigen Vector of A

$\mu \rightarrow$ Ritz Value of v. $(\mu, v) \rightarrow$ Ritz pair

If (μ, v) is an eigenpair

then $Av - \mu v = 0$

else $\| Av - \mu v \|_2 \neq 0$.

but, close to zero if (μ, v) is good approximation of eigenpair of A.

On the other hand, if $\cancel{(\mu, v)}$ $\| Av - \mu v \|_2 \approx 0$

then good reason to expect that (μ, v) is close to be an eigenpair of A. (atleast if A is normal)