An $O(N^2)$ Algorithm for Computation of the Minimum Time Consensus

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***Abstract*— The problem of achieving minimum time consensus for an N-agent system, with double integrator agents having bounded inputs, is considered. At the initial time instant, each agent has access to the state information about all the other agents. An algorithm, of $O(N^2)$ complexity, is proposed to compute the final consensus target state and minimum time to achieve this consensus. Further, local control laws are synthesized to drive each agent to the target point in the computed minimum time to consensus.**

I. Introduction

In recent years, cooperative control of multi-agent systems has been one of the most active areas of research [1], [2]. This area has found an increasing number of applications such as unmanned aerial, underwater and ground vehicles [3], [4], [5]. One of the most studied problems in this area is that of designing distributed control laws for multi-agent systems to achieve consensus (see [2], [6] and references therein). The agents are commonly assumed to be single-integrators [7], [8] with position as the state variable or double-integrators [1], [9] with position and velocity as state variables.

One of the important design parameter for consensus laws is the speed of achieving consensus. Various consensus algorithms are proposed that converge asymptotically [7], [10]. In almost all such consensus algorithms, the speed of reaching consensus can be characterized by the algebraic connectivity of the communication graph [11]. As a result, various properties of the communication graph have been designed to maximize the algebraic connectivity so as to speed up the convergence to consensus [12], [13]. On the other hand, [14], [15] tackle the problem of computing control laws to achieve consensus in finite time. Yet, there are only few articles that consider the minimum time consensus problem [16], [17], [18].

In a recent paper [19], we introduced a method for computation of the time optimal consensus point for N double integrators with bounded input, based on Helly's theorem [20]. This method uses the convexity of attainable sets. In [19], the minimum time to consensus and the corresponding consensus point are computed explicitly for each triplet of the agents leading to a runtime complexity of $O(N^3)$. In this paper, we exploit the properties of *Reachable sets* to give an algorithm with $O(N^2)$ run time complexity. Using this algorithm, we compute the smallest reachable set that encloses the initial conditions of all the agents. This reachable set is characterized by the minimum time to consensus and corresponding target point. We obtain these necessary parameters from the proposed algorithm. This algorithm follows steps analogous to an algorithm proposed in [21] to obtain a solution to the classical problem of finding *Smallest Enclosing Circle [22]*. Similar to the method proposed in [19], our algorithm needs global information about states of each agent at the initial time instant. Once the optimal time and consensus state are identified we compute local strategies for each agent which steers the state trajectory of all agents to the consensus state exactly at the optimal time.

The common assumption of incomplete communication graph is usually required since it is difficult to have all-to-all real time data transfer. However, contrary to the real time control objective, in this paper, we compute the target (minimum time) consensus point, once, only at the initial time instant. For this purpose, it is reasonable to assume that the global state information can be routed to all the agents over any connected communication network. Recent experiences with Zigbee, or Wifi based networks on UAVs [23], [24], [25], along with standard routing algorithms, suggest that this is possible with the currently available technology.

The paper is organized as follows. In Section II we first formalize the problem and introduce preliminaries required. In Section III, we characterize some properties of the time optimal consensus point of N-agents using Helly's theorem. Section IV is devoted to the proposed algorithm to find the smallest enclosing reachable set i.e. the reachable set to target consensus point at minimum time to consensus. Once the minimum time consensus state and time are identified in Section V we provide local strategies for all agents to steer towards the corresponding consensus state in the computed optimal time.

II. Problem Formulation and Preliminaries

A. Problem formulation

Consider a multi-agent system consisting of N agents, $a_1, ..., a_N$, with identical double integrator dynamics given by $\ddot{x}_i(t) = u_i(t)$ for $i = 1, ..., N$. In the state-space, the motion of each agent is governed by

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t),\ \mathbf{x}_i(0) = \mathbf{x}_{i0},\ \ i = 1, \cdots, N \quad (1)$$

where $\mathbf{x}_i(t) = \begin{bmatrix} x_i(t) & \dot{x}_i(t) \end{bmatrix}^T \in \mathbb{R}^2$ are the states (position and velocity respectively), $u_i(t) \in \mathbb{R}$ (acceleration) is the input, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. The initial condition of a_i is specified to be $\mathbf{x}_{i0} = \begin{bmatrix} x_{i0} & \dot{x}_{i0} \end{bmatrix}^T$. We assume that the input $u_i(t)$ of a_i is constrained to lie in the set $U(w_i) := \{u_i(t) : |u_i(t)| \le w_i \le 1\}$.

A multi-agent system is said to achieve *consensus* if for all $i,\ j \in \{1, \cdots, N\}$, $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \to 0$ as $t \to \bar{t} > 0$ and $\mathbf{x}_i(t) = \mathbf{x}_j(t)$ for all $t \ge \bar{t}$. The time $\bar{t}$ is called the *time*

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to consensus and the point $\mathbf{x}_i(\bar{t}) = \bar{\mathbf{x}}$ $(i = 1, ..., N)$ is the corresponding *consensus point*. The consensus is said to be achieved in finite time if $\bar{t} < \infty$.

Our objective is to identify a consensus point $\bar{\mathbf{x}} \in \mathbb{R}^2$ for a given set of initial conditions of the agents, such that the time to consensus $\bar{t}$ is minimum. Thus the minimum time consensus problem can be stated as

Problem 1. For an N-agent system (1) with $u_i(t) \in U(1)$ for $i = 1, ..., N$, find $\bar{\mathbf{x}}$ and $\bar{t} := \min\ t$ such that for all $i, j \in \{1, \cdots, N\}$, $\mathbf{x}_i(t) = \mathbf{x}_j(t)$ for $t \geq \bar{t}$.

Once the minimum time consensus point $\bar{\mathbf{x}}$ and corresponding time to consensus $\bar{t}$ are identified, we design local control law for each agent which drives the agent to the consensus point $\bar{\mathbf{x}}$ in time $\bar{t}$.

B. Preliminaries

In this section, since all the agents are identical LTI systems given by (1), we drop the subscript i for simplicity of notation. The set of all the states that an agent can reach from $\mathbf{p} \in \mathbb{R}^2$ using admissible control $u(t) \in U(w)$ in time $t > 0$ is called the *attainable set* ($\mathcal{A}_\mathbf{p}(t)$) from $\mathbf{p}$ *at* time t [27]. This set is given by,

$$\mathcal{A}_\mathbf{p}(t) = \left\{ \mathbf{x} : \mathbf{x} = e^{At}\mathbf{p} + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau,\ \forall u(t) \in U(w) \right\} \quad (2)$$

The set of initial conditions in $\mathbb{R}^2$ from which an agent can reach a point $\mathbf{p}$ in time t, using admissible control $u(t) \in U(w)$ is defined as the *reachable set* to a point $\mathbf{p}$ at time t [27],

$$R_\mathbf{p}(t) = \left\{ \mathbf{x} : \mathbf{x} = e^{-At}\mathbf{p} - \int_0^t e^{-A\tau} B u(\tau) d\tau,\ \forall u(t) \in U(w) \right\} \quad (3)$$

For a given $\mathbf{p}$ and t, sets $\mathcal{A}_\mathbf{p}(t)$ and $R_\mathbf{p}(t)$ are related by $\mathcal{A}_\mathbf{p}(t) = (e^{At} - I)\mathbf{p} + e^{At} R_\mathbf{p}(t)$.

Lemma 2. *[27], [28] The set $R_\mathbf{p}(t)$ and hence $\mathcal{A}_\mathbf{p}(t)$ are compact convex sets with nonempty interior.*

For a double integrator system, the attainable set with $\mathbf{p} = [-2.5\ \ 1]^T$ at time $t = 2$ is shown in Figure 1. Let

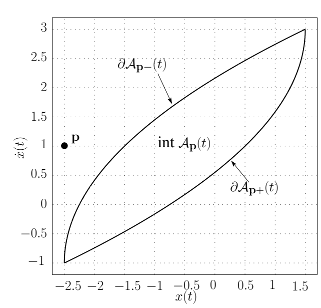

Fig. 1. $\mathcal{A}_\mathbf{p}(2)$ for a double-integrator with $\mathbf{p} = [\ -2.5 \quad 1\ ]^T$

$\partial\mathcal{A}_\mathbf{p}(t)$ and $\partial R_\mathbf{p}(t)$ denote the boundaries of $\mathcal{A}_\mathbf{p}(t)$ and $R_\mathbf{p}(t)$ respectively. From (2) and (3), it can be easily verified that [28] $\mathbf{q} \in \mathcal{A}_\mathbf{p}(t) \iff \mathbf{p} \in R_\mathbf{q}(t)$ Further,

$$\mathbf{q} \in \partial\mathcal{A}_\mathbf{p}(t) \iff \mathbf{p} \in \partial R_\mathbf{q}(t) \quad (4)$$

The following result is also taken from [28].

Theorem 3. *[28] Consider an LTI system (1) and a point $\mathbf{q} \in \mathbb{R}^2$. Let $u(t) \in U(w)$. The point $\mathbf{q} \in \partial R_\mathbf{p}(t)$ if and only if there exists a unique bang-bang control $\bar{u}(t)$ such that $|\bar{u}(t)| = w$ (with $\bar{u}(t)$ switching at-most once), that transfers the states of (1) from $\mathbf{q}$ to $\mathbf{p}$ exactly in time t.*

Hence, the boundary of a reachable set can be characterized as

$$\partial R_\mathbf{q}(t) = \left\{ \mathbf{x} : \mathbf{x} = e^{-At}\mathbf{q} - \int_0^t e^{-A\tau} B\bar{u}(\tau) d\tau \right\} \quad (5)$$

Since, $\mathbf{q} \in \partial\mathcal{A}_\mathbf{p}(t) \iff \mathbf{p} \in \partial R_\mathbf{q}(t)$, the boundary of an attainable set is given as

$$\partial\mathcal{A}_\mathbf{p}(t) = \left\{ \mathbf{x} : \mathbf{x} = e^{At}\mathbf{p} + \int_0^t e^{A(t-\tau)} B\bar{u}(\tau) d\tau \right\} \quad (6)$$

Following are the implicit expressions for the boundary of attainable set $\partial\mathcal{A}_\mathbf{p}(t)$, with $\mathbf{p} = [\ p \quad \dot{p}\ ]^T$. The boundary points starting with initial input $u(0) = \pm 1$ are given by the solutions to

$$\left(\pm x \mp t\dot{x} \mp p + \frac{t^2}{2} \right) = \left(\frac{\dot{x} - \dot{p} \mp t}{2} \right)^2 \quad (7)$$

We denote the interior of $\mathcal{A}_\mathbf{p}(t)$ by int($\mathcal{A}_\mathbf{p}(t)$).

III. Characterization of Time Optimal Consensus Point

In this section, we recall some of the important results from [19] for the sake of completeness of the presentation.

In a multi-agent system, with each agent a_i, we associate an attainable set from its initial condition $\mathbf{x}_{i0}$. We refer to the attainable set from $\mathbf{x}_{i0}$ at time t as the *attainable set of agent* a_i at time t and denote it by $\mathcal{A}_i(t)$. Clearly, for consensus to be possible at time $\bar{t}$, the intersection of the attainable sets for all the agents should be non-empty at $\bar{t}$, i.e. $\bigcap_{1 \leq i \leq N} \mathcal{A}_i(\bar{t}) \neq \phi$. It seems that checking this condition directly would require computing solution of a large set of coupled polynomial equations and inequalities simultaneously. However, as stated in Lemma 2, the attainable set $\mathcal{A}_i(t)$ of agent a_i at time t is a convex set in $\mathbb{R}^2$. Therefore, Helly's theorem on intersection of convex sets can be employed to simplify this computation.

Theorem 4. *(Helly's Theorem) [20] Let F be a finite family of at least $n+1$ convex sets in $\mathbb{R}^n$. If the intersection set of every $n+1$ members of F is nonempty, then all members of F have a nonempty intersection.*

Here, $n = 2$. Hence, if the intersection for each triplet of the attainable sets is nonempty at some time t, then the consensus can be achieved at time t. For minimum time consensus, we need to find minimum time $\bar{t}$ at which, the intersection of each triplet is nonempty.

For a triplet of agents, $\{a_i, a_j, a_k\}$, from N-agent system, let $\bar{t}_{ijk} := \min\ t$ such that $\mathcal{A}_i(t) \cap \mathcal{A}_j(t) \cap \mathcal{A}_k(t) \neq \phi\ \forall t \geq \bar{t}_{ijk}$. In other words,

$$\bar{t}_{ijk} := \min\ t \text{ such that } \mathbf{x}_i(t) = \mathbf{x}_j(t) = \mathbf{x}_k(t)\ \forall t \geq \bar{t}_{ijk}$$

Further, we denote the minimum time consensus point for the triplet $\{a_i, a_j, a_k\}$ by $\bar{\mathbf{x}}_{ijk}$, i.e. $\bar{\mathbf{x}}_{ijk} = \mathbf{x}_i(\bar{t}_{ijk}) = \mathbf{x}_j(\bar{t}_{ijk}) = \mathbf{x}_k(\bar{t}_{ijk})$.

In [19], authors have also shown that,

Theorem 5. *[19] For a multi-agent system with agent dynamics given by (1), the minimum time to consensus is* $\bar{t} = \bar{t}_{pqr} = \max_{1 \le i,j,k \le N} \bar{t}_{ijk}$ *and the consensus point* $\bar{\mathbf{x}} = \bar{\mathbf{x}}_{pqr}$ *is the minimum time consensus point of any triplet* $\{a_p, a_q, a_r\}$ *that achieves this maximum.*

We now characterize the minimum time consensus point of a triplet of agents.

Consider a triplet of agents $\{a_i, a_j, a_k\}$. For the attainable sets of these three agents to intersect, the attainable sets of every pair of agents must necessarily intersect. Clearly, $\bar{t}_{ijk} \ge \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$. Without loss of generality, let $\bar{t}_{ij} = \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$. Then, $\mathcal{A}_i(\bar{t}_{ij}) \cap \mathcal{A}_j(\bar{t}_{ij}) \ne \phi$, $\mathcal{A}_j(\bar{t}_{ij}) \cap \mathcal{A}_k(\bar{t}_{ij}) \ne \phi$ and $\mathcal{A}_i(\bar{t}_{ij}) \cap \mathcal{A}_k(\bar{t}_{ij}) \ne \phi$. Now for the third agent a_k, we have two cases namely 1) $\bar{\mathbf{x}}_{ij} \in \mathcal{A}_k(\bar{t}_{ij})$ and 2) $\bar{\mathbf{x}}_{ij} \notin \mathcal{A}_k(\bar{t}_{ij})$.

For both these cases, $\bar{\mathbf{x}}_{ijk}$ is characterized as

Lemma 6. *[19] If* $\bar{\mathbf{x}}_{ij} \in \mathcal{A}_k(\bar{t}_{ij})$*, then* $\bar{\mathbf{x}}_{ijk} = \partial\mathcal{A}_i(\bar{t}_{ij}) \cap \partial\mathcal{A}_j(\bar{t}_{ij}) = \bar{\mathbf{x}}_{ij}$ *and if* $\bar{\mathbf{x}}_{ij} \notin \mathcal{A}_k(\bar{t}_{ij})$*, then* $\bar{\mathbf{x}}_{ijk} = \partial\mathcal{A}_i(\bar{t}_{ijk}) \cap \partial\mathcal{A}_j(\bar{t}_{ijk}) \cap \partial\mathcal{A}_k(\bar{t}_{ijk})$.

IV. Smallest Enclosing Reachable Set (*SERS*) Algorithm

As shown in Theorem 5, the minimum time consensus point for an N-agent system, $\bar{\mathbf{x}}$ is the minimum time consensus point $\bar{\mathbf{x}}_{pqr}$ of a triplet $\{a_p, a_q, a_r\}$ such that $\bar{t}_{pqr} = \max_{1 \le i,j,k \le N} \bar{t}_{ijk}$. Thus, $\bar{\mathbf{x}}$ lies on the boundary of attainable sets of two or three agents. In other words,

Theorem 7. *The minimum time consensus point* $\bar{\mathbf{x}}$ *depends only on the initial condition of agents* a_i *for which* $\mathbf{x}_i(0) \in \partial R_{\bar{\mathbf{x}}}(\bar{t})$.

The proof follows from Lemma 6 and (4).

Further, as $\bar{\mathbf{x}} \in \cap_{i=1,..,N}\mathcal{A}_i(\bar{t})$, the initial conditions for each agent a_i, $\mathbf{x}_{i0} \in R_{\bar{\mathbf{x}}}(\bar{t})$. As a result, $\bar{\mathbf{x}}$ and $\bar{t}$, characterize the smallest reachable set $R_{\mathbf{x}}(t)$ (over parameters $\mathbf{x}$ and t) that encloses initial conditions of all the agents. We call this set the *Smallest Enclosing Reachable Set* (SERS).

A. Computing SERS

In [21], an $O(N^2)$ algorithm was proposed as a solution to the Smallest Enclosing Circle Problem [22]. For ease of notation, we refer this algorithm as SECA (Smallest Enclosing that is analogous to SECA. First, we define analogous terms to those used in SECA. Similar to SECA, the SERS algorithm also utilizes the fact that the initial conditions of either two or three agents lie on the boundary of the smallest enclosing reachable set (see Theorem 7). The initial conditions of the rest of the agents may lie either on the boundary or the interior of the reachable set. The **center** of the reachable set $R_{\mathbf{x}}(t)$ is the state $\tilde{\mathbf{x}}$ such that if a system has initial condition at $\tilde{\mathbf{x}}$, then, input $u(t) = 0$ drives the state trajectory of the system to $\mathbf{x}$ exactly at t. Thus, $\tilde{\mathbf{x}} = e^{-At}\mathbf{x}$. A **diameter** of $R_{\mathbf{x}}(t)$ is the line $\mathbf{x}_{i0}\mathbf{x}_{j0}$, joining the initial conditions $\mathbf{x}_{i0}$ and $\mathbf{x}_{j0}$ of the agents a_i and a_j respectively such that, $\mathbf{x}$ is the minimum time consensus point of a_i and a_j. It can be verified that, for any diameter of $R_{\mathbf{x}}(t)$, the center of $R_{\mathbf{x}}(t)$, $\tilde{\mathbf{x}} = \frac{\mathbf{x}_{i0}+\mathbf{x}_{j0}}{2}$.

Proposition 8. *Consider three agents* a_1, a_2 *and* a_3 *such that* $\mathbf{x}_{i0} \in R_{\mathbf{x}}(t)$ *for* $i = 1, 2, 3$*. If the center of* $R_{\mathbf{x}}(t)$*,* $\tilde{\mathbf{x}} \in$ *ConvHull*$\{\mathbf{x}_{10}, \mathbf{x}_{20}, \mathbf{x}_{30}\}$ *then time* $\bar{t}_{123} = t$ *and* $\bar{\mathbf{x}}_{123} = \mathbf{x}$*. Conversely if* $\tilde{\mathbf{x}} \notin$ *ConvHull*$\{\mathbf{x}_{10}, \mathbf{x}_{20}, \mathbf{x}_{30}\}$*, then* $\bar{t}_{123} < t$.

The proof of Proposition 8 is omitted due to space constraint.

Next we propose an algorithm to compute the SERS. This algorithm takes initial conditions of agents $\mathbf{x}_{i0} \in \mathbb{R}^2$ as input and gives $\bar{\mathbf{x}}$ and $\bar{t}$ i.e., the minimum time consensus state and the minimum time required to reach it as output. Using the example of a multi-agent system with 6 agents $\{a_1, \cdots, a_6\}$, we simultaneously demonstrate how the algorithm works.

Example 9. Let the initial conditions of these agents be $\mathbf{x}_{10} = \begin{bmatrix} -2.08 & 1.08 \end{bmatrix}^T$, $\mathbf{x}_{20} = \begin{bmatrix} -1.1 & 0 \end{bmatrix}^T$, $\mathbf{x}_{30} = \begin{bmatrix} -1 & -1 \end{bmatrix}^T$, $\mathbf{x}_{40} = \begin{bmatrix} -1 & 0.25 \end{bmatrix}^T$, $\mathbf{x}_{50} = \begin{bmatrix} -1.5 & -0.5 \end{bmatrix}^T$ and $\mathbf{x}_{60} = \begin{bmatrix} -2.5 & 1 \end{bmatrix}^T$ respectively.

According to Theorem 7, there are two or three agents on the boundary of the smallest enclosing reachable set. In this algorithm, we try to find these agents iteratively. We denote a guess of boundary agents by $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$. The initial conditions of these agents are denoted by $\boldsymbol{x}_1, \boldsymbol{x}_2$ and $\boldsymbol{x}_3$ respectively.

Smallest Enclosing Reachable Set (SERS) Algorithm

Our first guess the minimum time consensus point $\hat{\mathbf{x}}_0$ is the origin. Observe that, for any t, the center of $R_{\hat{\mathbf{x}}_0}(t)$ is $\tilde{\mathbf{x}}_0 = \hat{\mathbf{x}}_0$.

Step 1: Find $\mathbf{x}_{i0}$ such that minimum time required for a_i to reach origin is the largest among all the agents. Let $\hat{t}_0$ denote the minimum time required to drive the states of a_i from $\mathbf{x}_{i0}$ to origin. Our first guess of SERS is $R_{\hat{\mathbf{x}}_0}(\hat{t}_0)$. We set $\mathbf{a}_1 = a_i$ and $\boldsymbol{x}_1 = \mathbf{x}_{i0}$.

In Example 9, $\mathbf{a}_1 = a_3$ with $\boldsymbol{x}_1 = \mathbf{x}_{30} = \begin{bmatrix} -1 & -1 \end{bmatrix}^T$ and $\hat{t}_0 = 3.4495$. As seen in the Figure 2 the set $R_{\hat{\mathbf{x}}_0}(\hat{t}_0)$

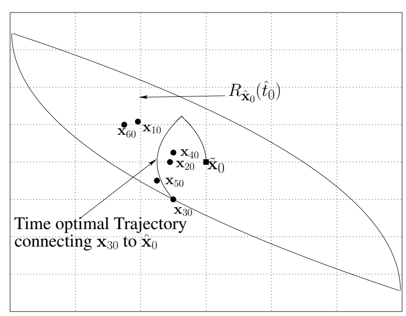

Fig. 2. SERS-Step 1

encloses all other agents' initial condition while $\boldsymbol{x}_1 = \mathbf{x}_{30} \in \partial R_{\hat{\mathbf{x}}_0}(\hat{t}_0)$.

Step 2: The next guess of the minimum time consensus point, $\hat{\mathbf{x}}_1$ is a point on the state trajectory joining $\boldsymbol{x}_1$ and the origin such that a new point $\mathbf{x}_{j0} \in \partial R_{\hat{\mathbf{x}}_1}(\hat{t}_1)$. This step is based on Theorem 3. We keep the switching time of the trajectory from $\boldsymbol{x}_1$ to the origin unchanged and find a point $\hat{\mathbf{x}}_1$ and time $\hat{t}_1 < \hat{t}_0$ such that some agent a_j can reach $\hat{\mathbf{x}}_1$ at $\hat{t}_1$ using bang-bang control with one switch. Note that according to Theorem 3, $\mathbf{x}_{j0} \in \partial R_{\hat{\mathbf{x}}_1}(\hat{t}_1)$. Such an agent is guaranteed to exist due to continuity of $\partial R_{\mathbf{x}}(t)$ in t and $\mathbf{x}$ [28]. We set $\boldsymbol{x}_2 = \mathbf{x}_{j0}$ and $\mathbf{a}_2 = a_j$. The center of $R_{\hat{\mathbf{x}}_1}(\hat{t}_1)$ is denoted by $\tilde{\mathbf{x}}_1 = e^{-A\hat{t}_1}\hat{\mathbf{x}}_1$.

In Example 9, $\mathbf{a}_2 = a_4$ i.e.. $\boldsymbol{x}_2 = \mathbf{x}_{40} = \begin{bmatrix} -1 & 0.25 \end{bmatrix}^T$. The next guess of the target point is $\hat{\mathbf{x}}_1 = \begin{bmatrix} -0.6073 & 1.103 \end{bmatrix}^T$ and the time to consensus as $\hat{t}_1 = 2.3473$ (see Figure 3).

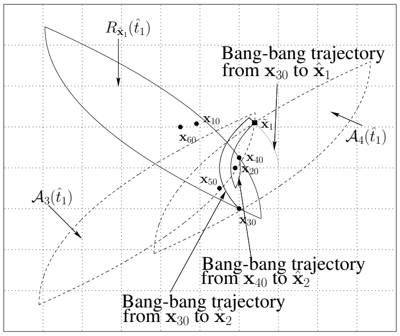

Fig. 3. SERS-Step 2

Step 3: If $\tilde{\mathbf{x}}_1 = \frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}$, i.e. $\boldsymbol{x}_1\boldsymbol{x}_2$ is the diameter of the reachable set $R_{\hat{\mathbf{x}}_1}(\hat{t}_1)$, then according to Theorem 7, $\bar{t} = \hat{t}_1$ and $\bar{\mathbf{x}} = \hat{\mathbf{x}}$. Otherwise CONTINUE to step 4. Here onwards, the algorithm is iterative. Hence, we denote the guesses of the target point, minimum time to consensus and the center of reachable set by $\hat{\mathbf{x}}_m, \hat{t}_m$ and $\tilde{\mathbf{x}}_m$ respectively. Value of m is updated after each iteration.

In Example 9, $\tilde{\mathbf{x}}_1 \neq \frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}$. Thus, we go to the next step.

Step 4: If $\tilde{\mathbf{x}}_m \neq \frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}$, then consider attainable sets $\mathcal{A}_{\boldsymbol{x}_1}(\hat{t}_m)$ and $\mathcal{A}_{\boldsymbol{x}_2}(\hat{t}_m)$. Since $\boldsymbol{x}_1\boldsymbol{x}_2$ is not a diameter of $R_{\tilde{\mathbf{x}}_m}(\hat{t}_m)$, we have $|\partial\mathcal{A}_{\boldsymbol{x}_1}(\hat{t}_m) \cap \partial\mathcal{A}_{\boldsymbol{x}_2}(\hat{t}_m)| = 2$. Consider a locus of points defined by $\partial\mathcal{A}_{\boldsymbol{x}_2}(t) \cap \partial\mathcal{A}_{\boldsymbol{x}_2}(t)$ for $0 \le t < \hat{t}_m$, which is a pair of points parametrized by t (since the points on the boundary $\partial\mathcal{A}_{\boldsymbol{x}_1}(t)$ and $\partial\mathcal{A}_{\boldsymbol{x}_2}(t)$ are solutions to quadratic equation (7) with appropriate signs. Let us denote the locus by $L(t)$. The next guess of target point is $\hat{\mathbf{x}}_{m+1} \in L(\hat{t}_{m+1})$ with $\hat{t}_{m+1} < \hat{t}_m$ such that either

(a) $|\partial\mathcal{A}_{\boldsymbol{x}_1}(\hat{t}_{m+1}) \cap \partial\mathcal{A}_{\boldsymbol{x}_2}(\hat{t}_{m+1})| = 1$ in which case $\hat{\mathbf{x}}_{m+1} \in \partial\mathcal{A}_{\boldsymbol{x}_1}(\hat{t}_{m+1}) \cap \partial\mathcal{A}_{\boldsymbol{x}_2}(\hat{t}_{m+1})$ and the line $\boldsymbol{x}_1\boldsymbol{x}_2$ forms a diameter of $R_{\hat{\mathbf{x}}_{m+1}}(\hat{t}_{m+1})$ or
(b) there is a third agent a_k such that $\mathbf{x}_{k0} \in \partial R_{\hat{\mathbf{x}}_{m+1}}(\hat{t}_{m+1})$.

The guess $\hat{t}_{m+1}$ can be found by solving the equations of $\partial\mathcal{A}_{\boldsymbol{x}_1}(t)$ and $\partial\mathcal{A}_{\boldsymbol{x}_2}(t)$ (i.e. equation (7)) and the solution of (1) for a_k using bang-bang control with one switch. If no such a_k exists, we get (a).

If (a) occurs then $\bar{\mathbf{x}} = \hat{\mathbf{x}}_{m+1}$ and $\bar{t} = \bar{t}_{m+1}$. Also if (b) occurs we set $\mathbf{a}_3 = a_k$ and $\boldsymbol{x}_3 = \mathbf{x}_{k0}$. If the center $\tilde{\mathbf{x}}_{m+1} = e^{-A\hat{t}_{m+1}}\hat{\mathbf{x}}_{m+1} \in \text{ConvHull}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$, then, $\bar{t} = \hat{t}_{m+1}$ and $\bar{\mathbf{x}} = \hat{\mathbf{x}}_{m+1}$ (see Proposition 8). If any of these conditions occur we stop, otherwise CONTINUE to the next step.

In Example 9, we move along the locus of intersection of the boundary of attainable sets $\partial\mathcal{A}_{\boldsymbol{x}_1}(t)$ and $\partial\mathcal{A}_{\boldsymbol{x}_2}(t)$ for $t < \hat{t}_2 = 2.3474$ till we obtain a new agent a_6 with initial condition lying on the boundary of $\mathbf{x}_{60} \in R_{\hat{\mathbf{x}}_2}(\hat{t}_2)$ with $\hat{\mathbf{x}}_2 = \begin{bmatrix} -1.4395 & -0.0625 \end{bmatrix}^T$ and $\hat{t}_2 = 1.5625$. We set $\mathbf{a}_3 = a_6$ and $\boldsymbol{x}_3 = \mathbf{x}_{60} = \begin{bmatrix} -2.5 & 1 \end{bmatrix}^T$. Now, we have $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in R_{\hat{\mathbf{x}}_2}(\hat{t}_2)$ (See Figure 4). Here, $\tilde{\mathbf{x}}_2 \in \text{ConvHull}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ and hence $\bar{\mathbf{x}} = \hat{\mathbf{x}}_2 = \begin{bmatrix} -1.4395 & -0.0625 \end{bmatrix}^T$ and $\bar{t} = \hat{t}_2 = 1.5625$.

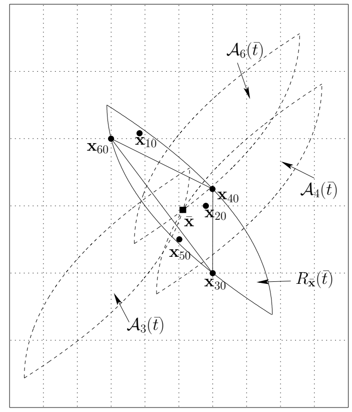

Fig. 4. SERS-Step 3

Step 5: If center $\tilde{\mathbf{x}}_{m+1} = e^{-A\hat{t}_{m+1}}\hat{\mathbf{x}}_{m+1} \notin \text{ConvHull}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$, then we renumber the boundary agents such that $\mathcal{A}_{\boldsymbol{x}_1}(\hat{t}_{m+1}) \cap \mathcal{A}_{\boldsymbol{x}_2}(\hat{t}_{m+1}) \subset \mathcal{A}_{\boldsymbol{x}_3}(\hat{t}_{m+1})$. It can be shown that, if $\mathcal{A}_{\boldsymbol{x}_1}(\hat{t}_{m+1}) \cap \mathcal{A}_{\boldsymbol{x}_2}(\hat{t}_{m+1}) \subset \mathcal{A}_{\boldsymbol{x}_3}(\hat{t}_{m+1})$ is satisfied at time $\hat{t}_{m+1}$ then for all time $t < \hat{t}_{m+1}$ we have $\mathcal{A}_{\boldsymbol{x}_1}(t) \cap \mathcal{A}_{\boldsymbol{x}_2}(t) \subset \mathcal{A}_{\boldsymbol{x}_3}(t)$. Thus, we ignore the initial condition $\boldsymbol{x}_3$ of the agent $\mathbf{a}_3$ from any further computations, set $m = m + 1$ and GOTO Step 3.

From Step 5, it can be seen that after each iteration one agent is removed from further iterations. Hence, the algorithm stops at p^{th} iteration ($p < N$) and we recover parameters $\bar{\mathbf{x}}$ and $\bar{t}$ of the minimum enclosing reachable set.

Remark 10. For a triplet $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ such that, $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \partial R_{\mathbf{x}}(t)$ for some $\mathbf{x}$ and t. Let $\bar{t}_{ij}$ be the minimum time to consensus for $\mathbf{a}_i$ and $\mathbf{a}_j$ for $i, j = 1, 2, 3$ and $i \neq j$. Let $\bar{t}_{12} = \max\{\bar{t}_{12}, \bar{t}_{13}, \bar{t}_{23}\}$. Then we can show that, for $t > \bar{t}_{12}$ either, $\mathcal{A}_{\boldsymbol{x}_1}(t) \cap \mathcal{A}_{\boldsymbol{x}_2}(t) \cap \mathcal{A}_{\boldsymbol{x}_3}(t) = \{\mathbf{x}\}$ or $\mathcal{A}_{\boldsymbol{x}_1}(t) \cap \mathcal{A}_{\boldsymbol{x}_2}(t) \subset \mathcal{A}_{\boldsymbol{x}_2}(t)$. Thus, in Step 5, $\mathcal{A}_{\boldsymbol{x}_1}(\hat{t}_{m+1}) \cap \mathcal{A}_{\boldsymbol{x}_2}(\hat{t}_{m+1}) \subset \mathcal{A}_{\boldsymbol{x}_3}(\hat{t}_{m+1})$ is guaranteed to happen.

Remark 11. The steps in SERS algorithm are analogous to steps in SECA, which in worst case, has run time complexity of $O(N^2)$. The number of computations at each step are fixed and hence the worst case run-time complexity of SERS algorithm is of same order as that of SECA i.e. $O(N^2)$.

V. Computation of Local Control Laws

Once the minimum time consensus point $\bar{\mathbf{x}}$ and corresponding time to consensus $\bar{t}$ are computed as discussed in Section III, we require a local control strategy $u_i(t)$ for each agent a_i, which will drive the states from $\mathbf{x}_{i0}$ to $\bar{\mathbf{x}}$ *exactly* at time $\bar{t}$. In this section, we suggest a method for the same.

To drive the states of an agent a_i from $\mathbf{x}_{i0}$ to $\bar{\mathbf{x}}$, first, we will synthesize bang-bang controls that switch between the extreme values of $U(w_i)$ (i.e. $\pm w_i$) according to some *switching surfaces* in the state-space [29], [30], [31]. Since we are concerned with single agent here, we drop the subscript i in this section.

The set of states that can be driven to $\bar{\mathbf{x}}$ using bang-bang input with 0 switches is given by,

$$M^{\pm}_{\bar{\mathbf{x}},0} = \{\mathbf{x} : \dot{x}^2 - \dot{\bar{x}}^2 \pm 2w(\bar{x} - x) = 0, \, \dot{x} \le \dot{\bar{x}}\} \tag{8}$$

The sets $M^{\pm}_{\bar{\mathbf{x}},0}$ act as switching surfaces. Thus, control input is obtained as

$$u(t) = w\text{sign}\left(\text{sign}(\dot{x} - \dot{\bar{x}})\left[\dot{x}^2 - \dot{\bar{x}}^2 + \text{sign}(\dot{x} - \dot{\bar{x}})2w(\bar{x} - x)\right]\right) \quad (9)$$

Case of initial conditions in $M^{+}_{\bar{\mathbf{x}},1} \cap M^{-}_{\bar{\mathbf{x}},1}$

The switching surfaces $M^{\pm}_{\bar{\mathbf{x}},0}$ are the sets of initial conditions that can be driven to $\bar{\mathbf{x}}$ using constant input $\pm w$. Observe that $M^{\pm}_{\bar{\mathbf{x}},0} \subset M^{+}_{\bar{\mathbf{x}},1} \cap M^{-}_{\bar{\mathbf{x}},1}$. When $\dot{\bar{x}} = 0$, the switching surfaces (8) meet smoothly at $\bar{\mathbf{x}}$ i.e. the derivatives of $M^{+}_{\bar{\mathbf{x}},0}$ and $M^{-}_{\bar{\mathbf{x}},0}$ are along the same line [31]. It can be easily verified that, in such case $M^{+}_{\bar{\mathbf{x}},1} \cap M^{-}_{\bar{\mathbf{x}},1} = M^{+}_{\bar{\mathbf{x}},0} \cup M^{-}_{\bar{\mathbf{x}},0}$ [31]. Hence there exists a unique (and hence time optimal [29]) bang-bang control that drives the state of the system to $\bar{\mathbf{x}}$. This is not the case when $\dot{\bar{x}} \neq 0$. In such cases, there can be some other points that lie in $M^{+}_{\bar{\mathbf{x}},1} \cap M^{-}_{\bar{\mathbf{x}},1}$. For these points, there exists multiple bang-bang controls with one switch that drive the states from $\mathbf{x}_0$ to $\bar{\mathbf{x}}$ but at different times. For example, the shaded region in Figure 5 is $M^{+}_{\bar{\mathbf{x}},1} \cap M^{-}_{\bar{\mathbf{x}},1}$. In this section, we analyse the trajectories

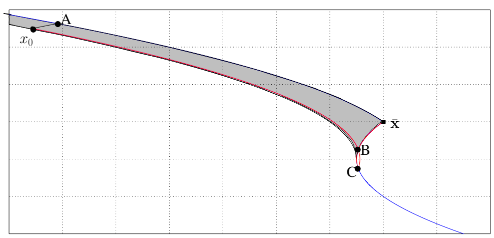

Fig. 5. Shaded region is $M^{+}_{\bar{\mathbf{x}},1} \cap M^{-}_{\bar{\mathbf{x}},1}$ of non-zero measure in $\mathbb{R}^2$

starting from $M^{+}_{\bar{\mathbf{x}},1} \cap M^{-}_{\bar{\mathbf{x}},1}$.

Let $\mathbf{x}_0$ be the initial condition of an agent. If the input $u(t) = \pm w$ is used, the states follow the trajectory

$$\mathbf{x}(t) = \begin{bmatrix} x_0 + t\dot{x}_0 \pm \frac{wt^2}{2} \\ \dot{x}_0 \pm wt \end{bmatrix} \quad (10)$$

that moves along the parabola

$$\dot{x}^2 - \dot{x}_0^2 \mp 2w(x - x_0) = 0 \quad (11)$$

From (9), it can be observed that, the switching surfaces are the portions of the parabolas (11) passing through $\bar{\mathbf{x}}$.

Lemma 12. *The initial condition $\mathbf{x}_0 \in M^{+}_{\bar{\mathbf{x}},1}$ (or $\mathbf{x}_0 \in M^{-}_{\bar{\mathbf{x}},1}$) if and only if (10) intersects $M^{-}_{\bar{\mathbf{x}},0}$ (or $M^{+}_{\bar{\mathbf{x}},0}$) at some point.*

Since the parabolas followed by the trajectories are symmetric about x-axis, we will discuss the case where $\dot{\bar{x}} \geq 0$. The case with $\dot{\bar{x}} < 0$ follows similarly.

Consider a case when $\dot{\bar{x}} > 0$. In this case, the switching surfaces meet at $\bar{\mathbf{x}}$, but the switching curve (defined by $M^{+}_{\bar{\mathbf{x}},0}$ when $\dot{x} \leq \dot{\bar{x}}$ and by $M^{-}_{\bar{\mathbf{x}},0}$ when $\dot{x} \geq \dot{\bar{x}}$) is not smooth at $\bar{\mathbf{x}}$ (i.e. their derivatives are not in the same direction). The corresponding parabolas intersect each other at two points.

Lemma 13. *Consider an initial conditions $\mathbf{x}_0$ such that*

$$\begin{aligned} -(\dot{x}_0^2 - \dot{\bar{x}}^2 - 2w(\bar{x} - x_0)) &< 0 \\ (\dot{x}_0^2 - \dot{\bar{x}}^2 + 2w(\bar{x} - x_0)) &> 0 \\ -(\dot{x}_0^2 + \dot{\bar{x}}^2 - 2w(\bar{x} - x_0)) &> 0 \end{aligned}$$

The trajectories (10) starting from $\mathbf{x}_0$ with $u(0) = -w$ intersect $M^{+}_{\bar{\mathbf{x}},0}$ at two points.

The $\mathbf{x}_0$'s satisfying the conditions of Lemma 5 are shown by shaded region in Figure 5.

Lemma 14. *For initial conditions that satisfy the conditions of Lemma 13, there exist three bang-bang controls that can drive the states to $\bar{\mathbf{x}}$ at different times.*

1) *Bang-bang control obtained using (9), i.e. using $u(t) = w$, until the trajectory hits $M^{-}_{\bar{\mathbf{x}},0}$ and then using $u(t) = -w$*
2) *Using $u(t) = -w$, until the trajectory hits $M^{+}_{\bar{\mathbf{x}},0}$ first time and then using $u(t) = -w$*
3) *Using $u(t) = -w$, until the trajectory hits $M^{+}_{\bar{\mathbf{x}},0}$ second time and then using $u(t) = -w$*

The trajectories generated by these bang-bang controls are demonstrated in Figure 5 by $\mathbf{x}_0\text{A}\bar{\mathbf{x}}$, $\mathbf{x}_0\text{B}\bar{\mathbf{x}}$ and $\mathbf{x}_0\text{C}\bar{\mathbf{x}}$. The input switches in these cases occur at A, B and C respectively.

Using principle of optimality, it can be shown that the first of these three bang-bang controls is the time-optimal control. In example demonstrated in Figure 5, the trajectory $\mathbf{x}_0\text{A}\bar{\mathbf{x}}$ reaches $\bar{\mathbf{x}}$ in time optimal manner at t^*.

There is another possibility where multiple bang-bang control exist to drive the states to $\bar{\mathbf{x}}$.

Lemma 15. *If the initial conditions $\mathbf{x}_0 \in M^{-}_{\bar{\mathbf{x}},0}$ (for the case $\dot{\bar{x}} > 0$), there exist two bang-bang controls that drive the states from $\mathbf{x}_0$ to $\bar{\mathbf{x}}$.*

Choosing appropriate bang-bang control

We need a control law by which it is possible to reach $\bar{\mathbf{x}}$ exactly at $\bar{t}$. Theorem 3, guarantees that there exists a bang-bang control with one switch that drives the state-trajectory from $\mathbf{x}_0$ to $\bar{\mathbf{x}}$ exactly at time $\bar{t}$ when $\mathbf{x}_0 \in \partial R_{\bar{\mathbf{x}}}(\bar{t})$. However, it need not be the time optimal control obtained using (9). Let $t^*(\mathbf{x})$ denote the minimum time required for state transfer from $\mathbf{x}_0$ to $\bar{\mathbf{x}}$.

Proposition 16. *The control input obtained using Algorithm 1, drives the states from $\mathbf{x}_0$ to $\bar{\mathbf{x}}$ at $\bar{t}$.*

The proof of this proposition is skipped due to space constraint. However, it can be verified from Figure 5.

Algorithm 1 Switching algorithm

Input: t, $\bar{t}$, $\mathbf{x}(t) = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$, $\bar{\mathbf{x}} = \begin{bmatrix} \bar{x} \\ \dot{\bar{x}} \end{bmatrix}$

$v(t)$ is the input obtained using (9).

$t^* = -\text{sign}(s)(\dot{\bar{x}} + \dot{x}) + \sqrt{\dot{\bar{x}}^2 + \dot{x}^2 + 4\text{sign}(s)(\bar{x} - x)}$

$t^*_s = \frac{t + \text{sign}(s)(\dot{\bar{x}} - \dot{x})}{2}$

if $\bar{t} - t == t^*$ **then** $u(t) = v$
    elseif $t^*_s == 0$ **then** $u(t) = v$
        else $u(t) = -v$
    endif
endif

Depending on whether $\bar{\mathbf{x}}$ lies on the boundary of the attainable set of an agent or in its interior, we use different values of w so that input obtained using Algorithm 1 can be used to drive the states to $\bar{\mathbf{x}}$ at $\bar{t}$. The computation of w is explained next.

1) For agents with $\bar{\mathbf{x}} \in \partial\mathcal{A}_i(\bar{t})$: For an agent a_i with $\mathbf{x}_{i0}$ and $\bar{\mathbf{x}} \in \partial\mathcal{A}_i(\bar{t})$, from (4), $\mathbf{x}_{i0} \in R_{\bar{\mathbf{x}}}(\bar{t})$. Thus, by Theorem 3, there is a unique bang-bang control law with one switch and $u_i(t) \in U(1)$. Thus, we use Algorithm 1 with full input limit $w_i = 1$.

2) For agents with $\bar{\mathbf{x}} \in int(\mathcal{A}_i(\bar{t}))$: Consider an agent a_i such that $\bar{\mathbf{x}} \in \mathrm{int}(\mathcal{A}_i(\bar{t}))$, i.e. $\mathbf{x}_{i0} \in \mathrm{int}(R_{\bar{\mathbf{x}}}(\bar{t}))$. We modify the notation of reachable set here to incorporate the input bounds. Let $R^w_{\mathbf{x}}(t)$ denote the reachable set of $\mathbf{x}$ at time t when $u(t) \in U(w)$. It is easy to verify that, $R^{w_1}_{\mathbf{x}}(t) \subsetneq R^{w_2}_{\mathbf{x}}(t)$ for $w_1 < w_2$. Thus, for given initial state $\mathbf{x}_{i0}$, final state $\bar{\mathbf{x}}$ and final time $\bar{t}$, we can choose w_i such that $\mathbf{x}_{i0} \in \partial R^{w_i}_{\bar{\mathbf{x}}}(\bar{t})$. In [19], authors have given closed form expressions to choose w_i which are the following. For input with $u_i(0) = \pm w_i$, we have $t_s = \frac{w_i\bar{t} \mp \nu_2}{2w_i}$ and

$$w_i = \mp\frac{\nu_2\bar{t} + 2\nu_1}{\bar{t}^2} + \frac{1}{\bar{t}^2}\sqrt{(\nu_2\bar{t} + 2\nu_1)^2 + (\nu_2\bar{t})^2}$$

The w_i for which $0 \le t_s \le \bar{t}$ is the required input bound. By appropriately choosing w_i, we can use Algorithm 1 to drive the agents state to $\bar{\mathbf{x}}$ at $\bar{t}$.

Example 17. For the multi-agent system defined in Example 9, the minimum time consensus point $\bar{\mathbf{x}} \in \mathrm{int}(\mathcal{A}_i(\bar{t}))$ for $i = 1, 2, 5$, while for $i = 3, 4, 6$, $\bar{\mathbf{x}} \in \partial\mathcal{A}_i(\bar{t})$. The agents a_3, a_4 and a_6 have to use time optimal control for reaching $\bar{\mathbf{x}}$ at $\bar{t}$. For remaining agents, the input constraints are $|u_1(t)| \le 0.8686$, $|u_2(t)| \le 0.4796$ and $|u_5(t)| \le 0.9057$. The simulation results obtained by using these control law are shown in Figure 6 which demonstrate that the minimum time consensus is indeed achieved.

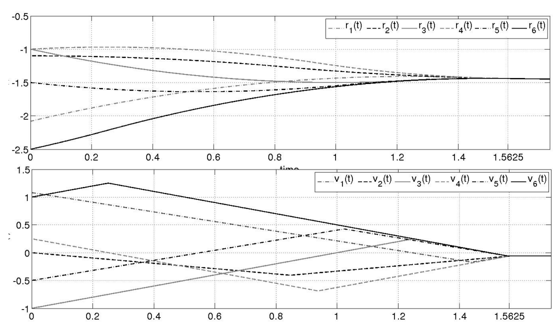

Fig. 6. Minimum time consensus

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