Iterative methods for $Ax = b$.

- x_0, guess (initial)
- generate x_1 from x_0
- x_2 from x_1 and so on,

until $\|b - Ax_i\|_2$ is sufficiently small.

- Disadvantage - would not work for multiple solutions.
- Advantage - Scalable to large matrices

- Jacobi's Method. - Non zero diagonal.

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

$$x_i = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j \right).$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)} \right).$$

$$x^{(k+1)} = D^{-1} \left[(D - A) x^{(k)} + b \right]$$

$$= x^{(k)} + D^{-1} \left[b - A x^{(k)} \right]$$

$$= x^{(k)} + D^{-1} r^{(k)}.$$

Iteration of type.

$$Mx^{(k+1)} = Nx^{(k)} + b \quad \text{—— } (*)$$

with.
M invertible

$[M = D, \text{ and } N = (D - A)]$, for jacobi method.

Requirements: for ease of implementation,
- M - non-singular
- easy to solve M.

for Rapid convergence.

$M \approx A$, $N \approx 0$

~~but~~ if $M \approx A$ then it violates easy to solve criterion.

Splitting: choose M & then $N = M - A$
~~For Jaco~~

At each step. $e^{(k)} = x - x^{(k)}$

Let $Mx = Nx + b$. $[Ax = b]$.

$$Me^{(k+1)} = Ne^{(k)} \Rightarrow e^{(k+1)} = M^{-1}Ne^{(k)}$$

$$e^{(k+1)} = G^{(k)}e^{(0)}$$

if $G^{(k)} \to 0$ then $e^{(k)} \to 0$

Choice of splitting crucial.

let $x^{(0)}$ be s.t. $e^{(0)} = b - Ax^{(0)}$

$e^{(0)} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

Where $v_1, v_2, \ldots, v_n$ are eigenvectors of G corresponding to $\lambda_1, \ldots, \lambda_n$.

$$e^{(1)} = Ge^{(0)} = c_1\lambda_1 v_1 + c_2\lambda_2 v_2 + \cdots + c_n\lambda_n v_n.$$

$$\vdots$$

$$e^{(k)} = G^{(k)}e^{(0)} = c_1\lambda_1^k v_1 + c_2\lambda_2^{(k)} v_2 + \cdots + c_n\lambda_n^k v_n.$$

$$\|e^{(k)}\|_2 \leq |c_1|\,|\lambda_1|^k\,\|v_1\|_2 + \cdots + |c_n|\,|\lambda_n|^k\,\|v_n\|_2$$

$|\lambda_i|^k \to 0$ as $k \to \infty$ if & only if

$|\lambda_i| < 1$

$\|e^{(k)}\|_2 \to 0$ for any initial guess $x^{(0)}$

if and only if $\max_i |\lambda_i| < 1$

Let $\sigma(G)$ be spectrum of G.

Spectral radius of G is max. distance of an eigenvalue from origin

$$\rho(G) := \max_{\lambda \in \sigma(G)} |\lambda|$$

The iterations converge iff $\rho(G) < 1$

and convergence is linear with convergence ratio $\rho(G)$

$$\left| \frac{\|e^{(k+1)}\|_2}{\|e^{(k)}\|_2} \right| \approx |\lambda_i| = \rho(G)$$

for sufficiently large k.

The smaller the G faster the convergence.

$$N = M - A \Rightarrow G = M^{-1} N = I - M^{-1} A$$

and $\rho(I - M^{-1}A) < 1$

Descent Methods.

- Steepest Descent

$$Ax = b.$$

Assume A symmetric & positive definite

$$J(y) = \frac{1}{2} y^T A y - y^T b$$

then minimizer of $J(y)$ is Solut. of $Ax = b$.

if A is positive definite then.
$\exists$ exactly one $x \in \mathbb{R}^n$. s.t
$J(x) = \min_y J(y)$

$$J(y) = \frac{1}{2} y^T A y - y^T b = \frac{1}{2} y^T A y - y^T A x$$

$$= \frac{1}{2} y^T A y - y^T A x + \frac{1}{2} x^T A x - \frac{1}{2} x^T A x$$

$$= \frac{1}{2} (y - x)^T A (y - x) - \frac{1}{2} x^T A x$$

Since A is positive definite
minimum only when $y = x$

- $\nabla J = Ay - b$.

$x^{(0)}, x^{(1)} \dots x^{(k)}$ s.t

$$J(x^{(k+1)}) \lneq J(x^{(k)})$$

$x^{(k)}$ to $x^{(k+1)}$.

- Choice of Search direction $p^{(k)}$
- a line search in chosen direction α_k.

Once Search direction $p^{(k)}$ is fixed

Choose ~~α from~~ $x^{(k+1)} \in \{x^{(k)} + \alpha p^{(k)} \mid \alpha \in \mathbb{R}\}$

typically α_k is chosen s.t.

$$J(x^{(k+1)}) = \min_{\alpha \in \mathbb{R}} J(x^{(k)} + \alpha p^{(k)})$$

Thm: $x^{(k+1)} = x^{(k)} + \alpha_k p^{(k)}$.

then $\quad \alpha_k = \dfrac{p^{(k)T} r^{(k)}}{p^{(k)T} A p^{(k)}}$

$$J(x^{(k)} + \alpha p^{(k)}) = \frac{1}{2}(x^{(k)} + \alpha p^{(k)})^T A (x^{(k)} + \alpha p^{(k)}) - (x^{(k)} + \alpha p^{(k)})^T b$$

$$= J(x^{(k)}) + \frac{1}{2}\alpha^2 p^{(k)T} A p^{(k)} - \alpha p^{(k)T}[b - Ax^{(k)}]$$

$$= J(x^{(k)}) + \frac{1}{2}\alpha^2 p^{(k)T} A p^{(k)} - \alpha p^{(k)T} r^{(k)} \quad \text{— (1)}$$

minimize for α ~~gives~~ $\alpha_k^* = p^{(k)T} r^{(k)} / p^{(k)T} A p^{(k)}$.

~~Step 1.~~ α_k is zero if & only if $[p^{(k)} \perp r^{(k)}$ or $r^{(k)} = 0$,].
Choose $p^{(k)} \not\perp r^{(k)}$ then $J(x^{(k+1)}) < J(x^{(k)})$
~~$J(y) = y^T A y + y^T b$~~.
(See (1).

Method works as ff.

(i) Initial guess:
$x^{(0)}$

(ii) Choose a direction not orthogonal to residual $r^{(0)} = b - A x^{(0)}$

$p^{(0)}$.

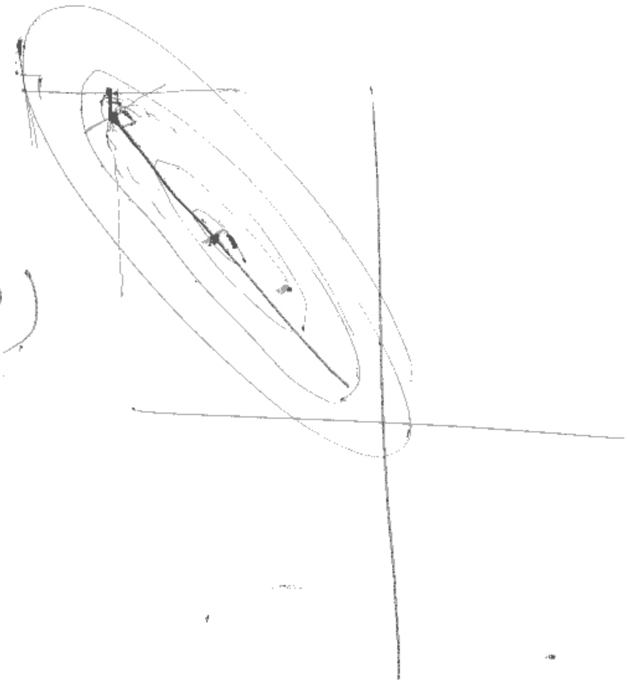
(iii) find $\alpha_0^* = \arg\min g(\alpha) := J(x^{(0)} + \alpha p^{(0)})$

(iv) $x^{(1)} = x^{(0)} + \alpha_0 p^{(0)}$.

Repeat.

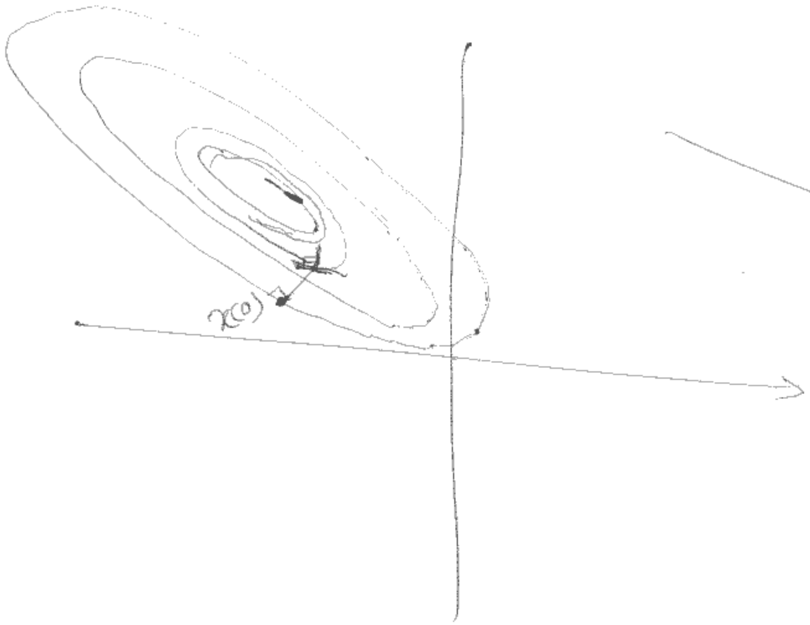
Steepest descent.

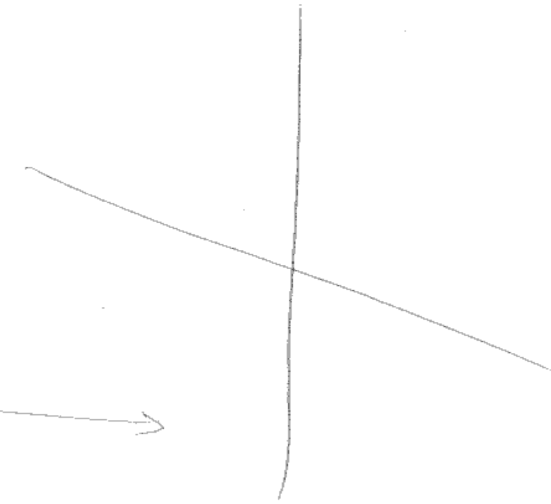
- Choose $p^{(k)} = r^{(k)}$
  $r^{(k)} = -\nabla J(x^{(k)})$



Contours of ~~constant~~ $J(y) = \text{Constant}$

form an ellipse



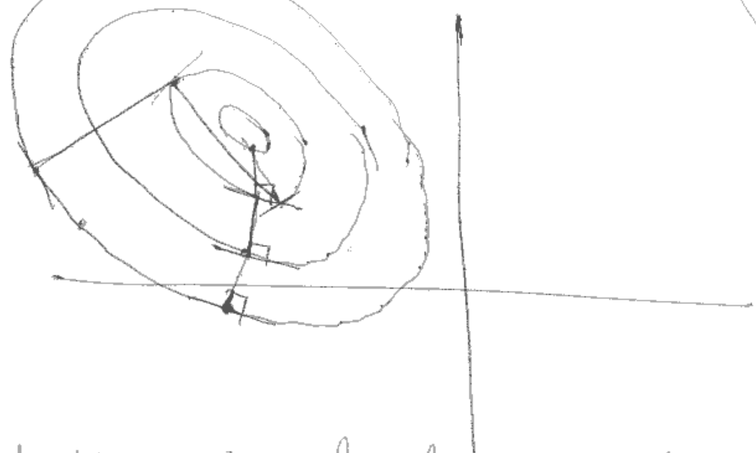


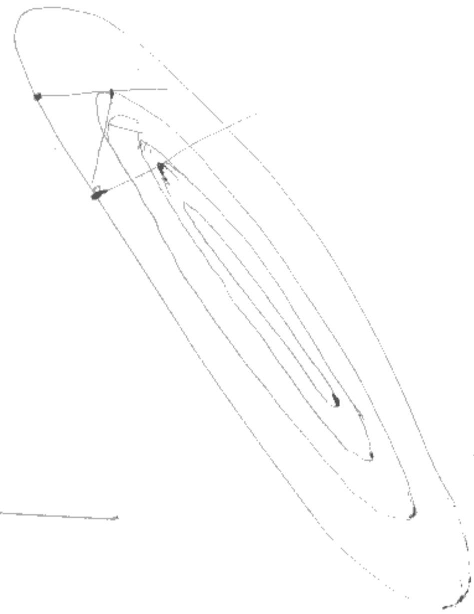
and gradient ∇J is perpendicular to each contour.

So when we start from $x^{(0)}$

and get $p^{(0)} = r^{(0)}$

$x^{(1)} = x^{(0)} + \alpha p^{(0)}$





Ill conditioning leads to slow convergence.

Iterative methods.

- Matrix Splitting methods — Jacobi iteration / Gauss Seidel iteration

Choose an initial guess x_0 & start iterating as.

$$Mx_{k+1} = Nx_k + b$$

where matrix A is split as.

$$A = M - N$$

with.

M - invertible. (easily)

Convergence guaranteed if $\rho(M^{-1}N) < 1$

- Descent methods. (Symmetric Positive def. A)

Unique minimizer of $J(y) = \frac{1}{2} y^T A y - y^T b$ is the solution.

~~Choose a direction~~

(i) Choose an initial guess x_0

(ii) Choose any direction p_0 s.t $p_0^T r_0 \neq 0$

$r_0 = (b - Ax^{0})$

(iii) Compute $\alpha_0 = \underset{\alpha \in R}{\arg\min}\, J(x_0 + \alpha p_0)$

(iv) ~~Next~~ update guess x_0 to

$x_1 = x_0 + \alpha_0 p_0$ and repeat.

When $p_0 = + r_0$ which is also gradient of $J(y)$

STEEPEST DESCENT METHOD.

Converges very slowly for ill-conditioned matrices