Q.2. Composition of two linear maps is equivalent to multiplication of matrices associated with them. In other words show that if

$f_1 : V_1 \to V_2$ and $f_2 : V_2 \to V_3$.

Let A_1 be matrix associated to f_1

and A_2 be matrix associated to f_2

then. $(f_2 \circ f_1) : V_1 \to V_3$

is represented by $A_2 \cdot A_1$.

Henceforth, we will always replace ~~assume~~ a matrix A ~~instead~~ for map $f : V_1 \to V_2$. Our main character has arrived

Lecture 3: Matrices.

From this class $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$

Let $A : V_1 \to V_2$ be a map from V_1 to V_2. Let $\dim V_1 = n$ $\dim V_2 = m$

A is also a matrix of size $m \times n$

We also say that $A \in \mathbb{R}^{m \times n}$.

and $A : \mathbb{R}^n \to \mathbb{R}^m$.

What A does is it takes a vector of n-components and gives out a vector of m components.

~~Set of~~

Given $A : \mathbb{R}^n \to \mathbb{R}^m$.

Image of A or Range of A or Column span of A is defined as.

$$\begin{array}{l}\mathrm{Im}\,A \\ \mathrm{Range}(A) \\ \mathrm{colspan}(A)\end{array} := \left\{ y \;\middle|\; \underset{\in \mathbb{R}^m}{y} = Ax \text{ for } x \in \mathbb{R}^n \right\}$$

So $\mathrm{Im}\,A$ is a subset of $\mathbb{R}^m$

Kernel of A or nullspace of A

$$\ker A := \left\{ \underset{\in \mathbb{R}^n}{x} \;\middle|\; Ax = 0 \right\}$$

Ker A is a subset of $\mathbb{R}^n$.

Given $A: V_1 \to V_2$

Inverse image of A or pre-image or $A^{-1}V_2$ is defined as:

$$A^{-1}V_2 := \{ x \in V_1 \mid Ax \in V_2 \}.$$

if $\dim V_1 = \dim V_2$ and inverse of A exists then it is same as matrix inverse that we already know of.

An easy hand calculation to ~~see~~ reveal kernel and image of matrix A is by doing column operations on A. Thinking about a matrix in terms of columns easily reveals a lot of information about it. Let us see this first.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & 8 \\ 7 & 8 & 9 & 14 \end{bmatrix}$$

Compute Ker A and Im A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & 8 \\ 7 & 8 & 9 & 14 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \rightarrow -2C_1 + C_2$

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 4 & -3 & 6 & 8 \\ 7 & -6 & 9 & 14 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \rightarrow -3C_1 + C_3, \quad C_4 \rightarrow -2C_1 + C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & -3 & -6 & 0 \\ 7 & 6 & -12 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \rightarrow -2C_2 + C_3$

$$A_{red} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 7 & 6 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\parallel$$

$$C_{ops.}$$

$$A\, C_{ops} = A_{red}.$$

$$A \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \hline \end{array}$$

$$A \left[\begin{array}{c|c|c|c} 1 & -2 & -3 & -2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 7 & 6 & 0 & 0 \end{bmatrix}$$

Now.

$$Av_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \quad Av_2 = \begin{bmatrix} 0 \\ -3 \\ 6 \end{bmatrix}, \quad Av_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Av_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now. $\ker A = \operatorname{span}\{ v_3, v_4 \}.$

$$\operatorname{im} A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 6 \end{bmatrix} \right\}.$$

Column operations reveal a ~~lot~~ complete picture about a matrix.