Lecture 3

1. Rank of a Matrix

number of linearly independent columns.

or

number of linearly independent rows

or

dim (image (A))

or dim (image (A^T))

2. $A: V_1 \to V_2$ Subspaces associated to A.

(i) Column space or image (A) or span of columns of A

(ii) Kernel or the nullspace of A (also called right nullspace)

(iii) Row space or image (A^T) or span of columns of A^T

(iv) Kernel of A^T or the (left nullspace of A)

3. $B = \{e_1, e_2, \cdots, e_n\}$

where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots$ $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

for $\mathbb{R}^n$ B is the standard basis ~~vector~~

span(B) = $\mathbb{R}^n$

Change of basis for a matrix is performing Similarity transformation

Let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear map expressed in a basis B_1 for $\mathbb{R}^n$

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\text{span}(B_1) \xrightarrow{A} \text{span}(B_1)$$

Let $B_1 = \{u_1, \dots, u_n\}$
~~and~~ $B = \{e_1, \dots, e_n\}$
$B_2 = \{v_1, \dots, v_n\}$

Suppose we want to change our point of view or the axes and represent A in terms of another Basis B_2

$$\text{Span}(B_2) \xrightarrow{A_{new}} \text{span}(B_2)$$

Express ~~B in~~ each vector in B in terms of basis B_1.

$$B = \left\{ \begin{aligned} e_1 &= l_{11}u_1 + l_{12}u_2 + \dots + l_{1n}u_n, \\ e_2 &= l_{21}u_1 + l_{22}u_2 + \dots + l_{2n}u_n, \\ &\vdots \\ e_n &= l_{n1}u_1 + l_{n2}u_2 + \dots + l_{nn}u_n \end{aligned} \right\} \quad [I = LU$$

let $l_1 = \begin{bmatrix} l_{11} \\ l_{12} \\ \vdots \\ l_{m1} \end{bmatrix}$ $l_2 = \begin{bmatrix} l_{21} \\ l_{22} \\ \vdots \\ l_{2n} \end{bmatrix}$ $\dots$ $l_n = \begin{bmatrix} l_{n1} \\ l_{n2} \\ \vdots \\ l_{nn} \end{bmatrix}$

$$B_1 \xrightarrow{L} B$$

Similarly by expressing B_2 in terms of B

$$B \xrightarrow{M} B_2$$

to change $u_1 \longrightarrow v_1$

$$u_1 \in B_1 \xrightarrow{L} e_1 \in B \xrightarrow{M} v_1 B_2$$

So,

$$v_1 = M L u_1$$

Now, Similarly

$$v_2 = M L u_2$$

$$\vdots$$

$$v_n = M L u_n$$

Let $ML = Q$

then.

$$\begin{tikzcd} B_1 \arrow[r, "A"] \arrow[d, "Q"'] & B_1 \\ B_2 \arrow[r, "A_{new}"] & B_2 \arrow[u, "Q^{-1}"'] \end{tikzcd}$$

$Au_1 =$ first column of A in basis B_1

$Au_2 =$ second column of A in basis B_1

$\vdots$

$Au_n =$ last column of A in basis B_1

Similarly, for A_{new} & B_2

$A_{new} v_1 =$ first column of A_{new}

~~$A u_1 =$~~ u_1 ~~$\longrightarrow Au_1$~~

$$\begin{tikzcd} \{u_1, \dots, u_n\} \arrow[r, "A"] \arrow[d, "Q"'] & \{Au_1, \dots, Au_n\} \\ \{v_1 \dots v_n\} \arrow[r, "A_{new}"] & \{A_{new} v_1 \dots A_{new} v_2\} \arrow[u, "Q^{-1}"'] \end{tikzcd}$$

any vector

$$\begin{tikzcd} {[x]_{B_1}} \arrow[r] \arrow[d, "Q"'] & A[x]_{B_1} \\ {[x]_{B_2}} \arrow[r] & A_{new}[x]_{B_2} \arrow[u, "Q^{-1}"'] \end{tikzcd}$$

$$A[x]_{B_1} = Q^{-1} A_{new} Q [x]_{B_1} \Leftrightarrow \boxed{A = Q^{-1} A_{new} Q}$$

Similarity Transformation

Change of basis is ~~leads~~ to Similarity transformation.

Example: $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & 2 \end{bmatrix} \longrightarrow A$

~~In general~~

$B = \{e_1, e_2, e_3\}$.

for a vector $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, suppose

We want to express A in basis

$B_{new} = \{x, Ax, A^2x\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} \right\}$.

$Ae_1 \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$Ae_2 \longrightarrow \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$Ae_3 \longrightarrow \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$.

$A_{new}\, x = \left\{ \text{first col. of } A_{new} \right.$

$A_{new}(Ax) =$ ~~first~~ Second col of A_{new}

$A_{new}(A^2x) =$ last col. of A_{new}

$$\begin{array}{ccc}
[x]_B & \xrightarrow{A} & A[x]_B \\
{\scriptstyle [x,\, Ax,\, A^2x] = Q}\ \big\downarrow & & \big\uparrow Q^{-1} \\
[x]_{B_1} & \xrightarrow{A_{new}} & A_{new}[x]_{B_1}
\end{array}$$

$A_{new} = A[x, Ax, A^2x]$

$= [Ax, A^2x, A^3x]$

$= [Ax, A^2x, 6x + 3A^2x]$

$= \begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}_{B_2}$

$Q = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 6 \end{bmatrix}, \quad Q^{-1} = ?$

Computational Notation.

Let us now clear up the notation we will be using for describing Computational algorithms

$$A \in \mathbb{R}^{m \times n} \iff A = (a_{ij}) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix}$$

with $a_{ij} \in \mathbb{R}$

1. Transposition

$$C = A^T \Rightarrow c_{ij} = a_{ji}$$

2. addition.

$$C = A + B \Rightarrow c_{ij} = a_{ij} + b_{ij}$$

3. Scalar-matrix multiplication

$$C = \alpha A \Rightarrow c_{ij} = \alpha a_{ij}$$

4. Matrix-Matrix multiplication ($\mathbb{R}^{m \times p} \times \mathbb{R}^{p \times n} \rightarrow \mathbb{R}^{m \times n}$)

$$C = AB \Rightarrow c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Vector notation :

$\mathbb{R}^n$ Vector space with real n-Vectors

$$x \in \mathbb{R}^n \iff x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad x_i \in \mathbb{R}.$$

(notice x is a column)

x_i is the i^{th} Component.

we may use $x(i)$ too Sometimes

x is also a $n \times 1$ matrix. By convention we will in this Course identify $\mathbb{R}^n$ with $\mathbb{R}^{n \times 1}$

$x \in \mathbb{R}^{1 \times n} \iff x = [x_1, \dots, x_n]$ which is a row.

If x is a column vector $y = x^T$ is row vector. So by transposition of x we can shift betn col^n & row vector.

Vector operations :

$\alpha \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n.$

$$z = \alpha x \Rightarrow z_i = \alpha x_i$$

$$z = x + y \Rightarrow z_i = x_i + y_i$$

dot product :

$$c = x^T y \Rightarrow c = \sum_{i=1}^{n} x_i y_i$$

vector multiply

$$z = x .* y \Rightarrow z_i = x_i y_i$$

<u>update form</u>: In many programs you update a given quantity by new one.

$y = \alpha x + y \Rightarrow y_i = \alpha x_i + y_i$

$\uparrow$

assignment in the sense of programming ~~Software~~ language.

① <u>Computation of Dot product.</u>

Algorithm 1: Compute a dot product.

Given $x, y \in \mathbb{R}^n$.

Compute $c = x^T y$.

initialize $c = 0$

for $i = 1:n$

$c = c + x(i)\,y(i)$

end.

This algorithm involves ~~two~~ n additions & n- multiplications.

Thus dot-product is $O(n)$ operations

② Matrix - Vector multiplication, updating a vector y

Suppose $A \in \mathbb{R}^{m\times n}$ & we wish to update $y = Ax + y$. Where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$

$$y_i = \sum_{j=1}^{n} a_{ij} x_j + y_i \qquad i = 1:m.$$

Algorithm : R.

for $i = 1 : m$

for $j = 1 : n$.

$y(i) = A(i, j)\, x(i) + y(i)$

end

end.

$\boxed{A(i, j) \rightarrow i^{th} \text{ row}, \; j^{th} \text{ column. entry.}}$

How many operations ?

(n multiplications) performed. (m) times

$O(mn)$

Notations :

$$A \in \mathbb{R}^{m \times n} \iff A = \begin{bmatrix} - & r_1^T & - \\ - & r_2^T & - \\ & \vdots & \\ - & r_m^T & - \end{bmatrix}, \quad r_k \in \mathbb{R}^n$$

row partitioning.

Column partitioning.

$$A \in \mathbb{R}^{m \times n} \iff A = \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix}, \quad c_i \in \mathbb{R}^m.$$

The Colon notation :

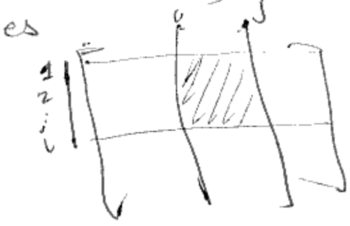
$A \in \mathbb{R}^{m \times n}$.

$A(k, :)$ is k^{th} row.

$A(:, k)$ is k^{th} column.

$A(1:i,\ i:j)$

gives



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for i = 1 : m.
    y(i) = A(i, :) x + y(i)
end.
```

Just like dot-product of two vectors (which is also called as inner product)

we also have an outer product.

$x y^T$ ~~$[x_1, x_2, x_3$~~ let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$$x y^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & & & \\ x_n y_1 & x_n y_2 & \cdots & x_n y_n \end{bmatrix}$$

Question: Write an algorithm to ~~for~~ update a given matrix A by outer product of x and y.

i.e perform $A = A + x y^T$

$$\underset{x}{\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}} \underset{y^T}{\begin{bmatrix} 1 & 6 \end{bmatrix}} = \begin{bmatrix} 1 & 6 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

$$C = AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$\cancel{C_{ij} = \sum a_{ij} b_{ji}} \quad \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_n^T \end{bmatrix}$$

$$(c_{ij}) = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right) = a_{i\cdot}^T b_{\cdot j} = A(i,:)^T B(:,j)$$

$$C = A \left[b_{\cdot 1}, b_{\cdot 2}, \ldots, b_{\cdot n} \right] = \left[A b_{\cdot 1} \quad A b_{\cdot 2} \quad \cdots \quad A b_{\cdot n} \right].$$

Outer product formulation:

$$C = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \end{bmatrix} + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} \begin{bmatrix} b_{21}, & b_{22}, & \cdots & b_{2n} \end{bmatrix} + \cdots$$

$$= a_1 b_1^T + a_2 b_2^T + \cdots + a_n b_n^T$$

n by n identity matrix

$$I_n = [e_1, \dots, e_n].$$

Where $e_k = [\underbrace{0, \dots, 0}_{k-1}, \underbrace{1}_{k\text{th entry}}, \underbrace{0, \dots, 0}_{n-k}]^T$

A and $X \in \mathbb{R}^{n \times n}$

$AX = I$, X is inverse of A.

$X = A^{-1}$.

if A^{-1} exists we say A is invertible.

$(AB)^{-1} = B^{-1}A^{-1}$.

$(AB)^{-1} X = I$.

$X = AB$

$B A^{-1} Y = B$

$B^{-1}A^{-1} Y = I$.

$X = B^{-1}A^{-1}$.

1. $\boxed{(A^{-1})^T = (A^T)^{-1}}$

2. $\boxed{B^{-1} = A^{-1} - B^{-1}(B-A)A^{-1}}$

3. $\boxed{(A + UV^T)^{-1} = A^{-1} - A^{-1}U(I + V^T A^{-1} U) V^T A^{-1}}$

The determinant :

$A = (a) \in R^{1\times 1}$

$\det(A) = a.$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \ddots & \\ a_{41} & \vdots & & \\ \vdots & & \cdots & a_{nn} \\ a_{n1} & a_{n2} & \cdots & \end{bmatrix}$$

$A \in \mathbb{R}^{n\times n}.$

$\cancel{\det(A) = \sum (-1)^{j+1} a_{ij} \det}$

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(A_{1j})$$

$(n-1)\times(n-1)$

A_{1j} is a square matrix obtained by deleting 1 row and j^{th} column.

$\det(AB) = \det A \det B \qquad A, B \in \mathbb{R}^{n\times n}$

$\det(A^T) = \det A. \qquad A \in \mathbb{R}^{n\times n}.$

$\det(cA) = c^n \det(A) \qquad c \in \mathbb{R}, \ A \in \mathbb{R}^{n\times n}.$

$\det(A) \neq 0 \iff A$ is non singular. $A \in \mathbb{R}^{n\times n}$

Suppose α is a scalar and. $A(\alpha)$ is m by n ①
Matrix with entries $a_{ij}(\alpha)$.

Let $a_{ij}(\alpha)$ be a differentiable function of
α for all i & j

then.

$$\dot{A}(\alpha) = \frac{d}{d\alpha} A(\alpha) = \left(\frac{d}{d\alpha} a_{ij}(\alpha)\right) = \left(\dot{a}_{ij}(\alpha)\right)$$