Lecture 4

To measure distances between two vectors / matrices is essential if we need to quantify errors in the results of algorith.

Vector norms

A vector norm on $\mathbb{R}^n$ is a function $f: \mathbb{R}^n \to \mathbb{R}$ that satisfies.

$$f(x) \geq 0 \qquad x \in \mathbb{R}^n \quad (f(x) = 0 \Leftrightarrow x = 0)$$

$$f(x+y) \leq f(x) + f(y) \qquad x, y \in \mathbb{R}^n$$

$$f(\alpha x) = |\alpha| f(x) \qquad \alpha \in \mathbb{R}^n, \; x \in \mathbb{R}^n.$$

$f(x) = \|x\|$ is usual notation.

Useful Class of norms:

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \dots + |x_n|^p\right)^{1/p}.$$

1, 2, ∞ are widely applicable & useful.

$$\|x\|_1 = \left(|x_1| + |x_2| + \dots + |x_n|\right)$$

$$\|x\|_2 = \left(|x_1|^2 + |x_2|^2 + \dots + |x_n|^2\right)^{1/2} = (x^T x)^{1/2}$$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

A unit vector w.r.t norm $\|\cdot\|$ is a vector x that satisfies $\|x\| = 1$

Holder inequality

$$|x^T y| \leq \|x\|_p \|y\|_q, \left(\frac{1}{p} + \frac{1}{q} = 1 \right)$$

special case.

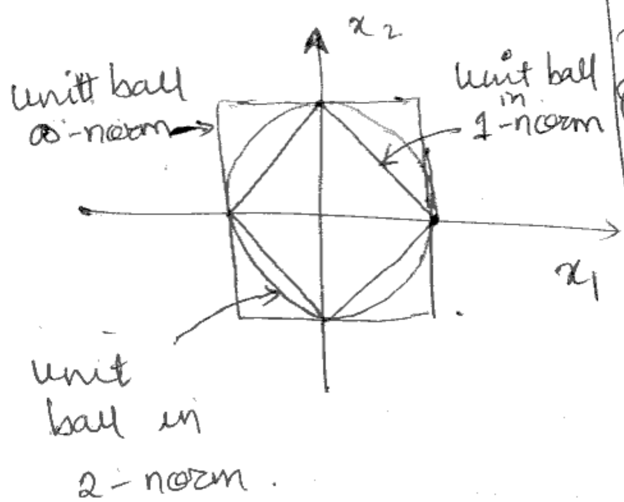
$$|x^T y| \leq \|x\|_2 \|y\|_2 .$$

Cauchy - Schwarz inequality.

All norms on $\mathbb{R}^n$ are equivalent.

$$c_1 \|x\|_\alpha \leq \|x\|_\beta \leq c_2 \|x\|_\alpha$$

Draw unit balls in $\mathbb{R}^2$ for ~~each~~ 1-norm, 2-norm, ∞-norm.



$|x_1| + |x_2| = 1.$

(1) $x_1 > 0, \ x_2 > 0.$

$x_1 + x_2 = 1$

(2) $x_1 < 0, \ x_2 > 0.$

$-x_1 + x_2 = 1.$

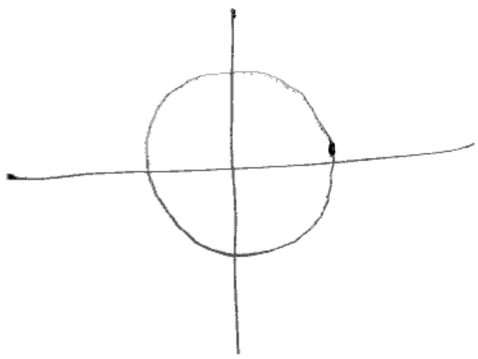
(3) $x_1 < 0 \quad x_2 < 0.$

$-x_1 - x_2 = 1$

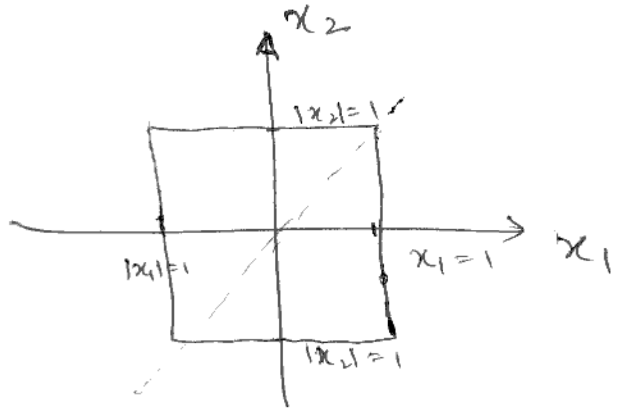
(4) $x_1 > 0, \quad x_2 < 0$

$x_1 - x_2 = 0$

② $x_1^2 + x_2^2 = 1$ ②



③ $\max\{|x_1|, |x_2|\} = 1.$



* Exercise try to plot unit ball for

$|x_1|^3 + |x_2|^3 = 1.$ ina matlab/scilab.

By a sequence $\{x^{(k)}\}$ we mean.

Vectors $x^{(0)}, x^{(1)}, x^{(2)}, \ldots, x^{(k)}, \ldots\ldots x^{(10000)} \ldots$

$\{x^{(k)}\}$.

This sequence converges to x if

$$\lim \| x^{(k)} - x \|_\alpha = 0$$

By equivalence of norms. we see that alpha does not matter $1 \le \alpha \le \infty$.

Cauchy Schwarz Inequality

$$|x^T y| \leq \|x\|_2 \, \|y\|_2 .$$

Proof: Consider for $t \in \mathbb{R}$

$$\|x + t y\|_2^2$$

$$= (x + t y)^T (x + t y)$$

$$= (x^T + t y^T)(x + t y)$$

$$= x^T x + 2t\, x^T y + t^2 y^T y$$

$$= c + bt + at^2 \geq 0$$

Where $c = x^T x$

$b = 2x^T y$

$a = y^T y.$

Since $c + bt + at^2 \geq 0$ for all real t | it cannot have two distinct real roots.

But it can have two same roots. inwhich case

$$b^2 - 4ac = 0$$

$$\Rightarrow \cancel{4}|x^T y|^2 = \cancel{4}\,(x^T x)(y^T y)$$

$$|x^T y| = (x^T x)^{1/2} (y^T y)^{1/2}$$

Or $b^2 - 4ac \leq 0.$

$$\Rightarrow \cancel{4}|x^T y| \leq \cancel{4}(x^T x)^{1/2} (y^T y)^{1/2}$$

Two norm Satisfies triangle inequality

$$\|x + y\|_2^2 \leq \|x\|_2^2 + \|y\|_2^2$$

$$(x+y)^T (x+y) = x^T x + 2x^T y + y^T y$$

$$= x^T x + 2x^T y + y^T y.$$

$$\leq (x^T x) + 2(x^T x)^{1/2} (y^T y)^{1/2} + (y^T y)$$

$$\leq (x^T x + y^T y)^2$$

$$\underline{\|x+y\|^2 \leq x^T x + y^T y}$$

$$|x+y| \leq |x| + |y|$$

One norm.

$$\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} (|x_i| + |y_i|) \leq \|x\|_1 + \|y\|_1$$

HW
Prove it for infinity norm.

Positive definite matrix:

A matrix ~~is~~ $A \in \mathbb{R}^{n \times n}$ is called positive definite

$$x^T A x \geq 0 \quad \forall x \in \mathbb{R}^n$$

and if $x^T A x = 0 \Rightarrow x = 0$] — ⊕

then it is called as "strictly" positive definite matrix

We will call A ~~use~~ positive semidefinite if ⊕ is not satisfied

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$$x^T A x = \frac{1}{2} x^T (A + A^T) x$$

$$= x^T \underbrace{\left[\frac{A + A^T}{2}\right]}x + \underbrace{x^T \left[\frac{A - A^T}{2}\right] x}_{= 0}$$

$$A = \underbrace{\frac{A + A^T}{2}}_{\text{symmetric part}} + \underbrace{\frac{A - A^T}{2}}_{\text{skew symmetric part}}$$

~~Thus~~ $\frac{A + A^T}{2}$ is positive definite

$\Leftrightarrow$ A is positive definite.

We only consider Symmetric matrices in our discussion about positive definite matrices i.e., In our scheme of things positive definiteness ~~matrix~~ is defined for symmetric matrices. We will see more about it later.

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(1) $f(x) = (x^T A x)^{1/2}$ is also a norm &

$\|x\|_A = (x^T A x)^{1/2}$.

for $A = I$

$\|x\|_A = \|x\|_2$.

Matrix Norms:

$A \in \mathbb{R}^{m \times n}$.

A ~~matrix~~ matrix norm is a function $f : \mathbb{R}^{n \times n} \to \mathbb{R}$

s.t.

1. $f(A) > 0$ if $A \neq 0$
2. $f(\alpha A) = |\alpha| f(A)$
3. $f(A + B) \leq f(A) + f(B)$
4. $f(AB) \leq f(A) f(B)$ (Submultiplicative property)

① Frobenius Norm.

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij})^2 \right)^{1/2}$$

Ⓠ Is $\|A\|_{max} = \max |a_{ij}|$ a norm?

~~Does not satisfy 4.!~~

$$\cancel{A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}}$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\|AB\|_{max} = 2 \geq \|A\|_{max} \|B\|_{max} = 2$$

8

No. It will satisfy 1, 2, 3

but not

~~f(A B)~~ $\|AB\|_{max} \leq \|A\|_{max} \|B\|_{max}$.

Consider

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

$$\left.\begin{array}{l} \|AB\|_{max} = 8 \\ \|A\|_{max} = 2 \\ \|B\|_{max} = 2 \end{array}\right\} \text{ but } \|AB\|_{max} > \|A\|_{max} \|B\|_{max}.$$

Not a matrix norm (Submultiplicative)

Although only properties 1, 2, 3 are needed for any function to be a norm. For matrices the fourth property is also ~~important~~ conventionally required to be satisfied for it to be a norm. (Some books may not specify it. It is very much author dependent!)

~~But~~

* Induced norms.

or Operator norms.

Remember Matrix is a linear transformation $T: V \to V$

Induced norms on matrices use norm on vector space to define a norm on the linear transformation.

(i) $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. [Linear operator / Linear map / Transformation]

$x \longmapsto Ax$

$\|x\|_v \quad \|Ax\|_v$

$$\|A\|_M = \max_{x \neq 0} \frac{\|Ax\|_v}{\|x\|_v}$$

Induced matrix norm / Operator norm.

(i) Thm: $\|Ax\| \leq \|A\| \, \|x\|$ — (1)

Induced norm.

(ii) $\|A\|$ is a matrix norm.

(i). $A = 0 \Rightarrow \|A\| = \max \dfrac{0}{\|x\|} = 0.$

assume $\|A\| = 0 \Rightarrow \displaystyle\max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|} = 0.$

$\Rightarrow \|Ax\| = 0$ for all $x \neq 0$.

$\Rightarrow A = 0$.

(2) $$\|\alpha A\| = \max_{\|x\|\neq 0} \frac{\|\alpha A x\|}{\|x\|}$$

$$= \max |\alpha| \frac{\|Ax\|}{\|x\|}$$

$$= |\alpha| \, \|A\|$$

3 $$\|A + B\| = \max_{\|x\|\neq 0} \frac{\|(A+B)x\|}{\|x\|}$$

$$= \max_{\|x\|\neq 0} \frac{\|Ax + Bx\|}{\|x\|}$$

$\cancel{\max(a+b)}$

$$\leq \max_{\|x\|\neq 0} \left(\frac{\|Ax\|}{\|x\|} + \frac{\|Bx\|}{\|x\|} \right)$$

$$\leq \|A\| + \|B\|$$

4 $$\|AB\| = \max_{\|x\|\neq 0} \frac{\|ABx\|}{\|x\|}$$

$$\leq \max \frac{\|A\| \, \|Bx\|}{\|x\|} \quad \text{by } 1$$

$$\leq \|A\| \, \|B\|$$

Induced norm is indeed a matrix norm.

induced 2- norm

$$\|A\|_2 = \max_{\|x\|_2 \neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

does magnitude of x play any role in $\|A\|_2$?

No.

Consider.

Let $x \neq 0$, $\cancel{cx \neq 0}$

be

$$\text{s.t } \|A\|_2 = \frac{c}{\cancel{c}} \frac{\|Ax\|_2}{\|x\|_2} = \cancel{\frac{\|A\,cx\|_2}{\|cx\|_2}}$$

$$\text{then } \cancel{\|A\|_2 =} \frac{\|A\,cx\|_2}{\|cx\|_2}$$

$$= \frac{|\cancel{c}|\,\|Ax\|_2}{|\cancel{c}|\,\|x\|_2} = \|A\|_2$$

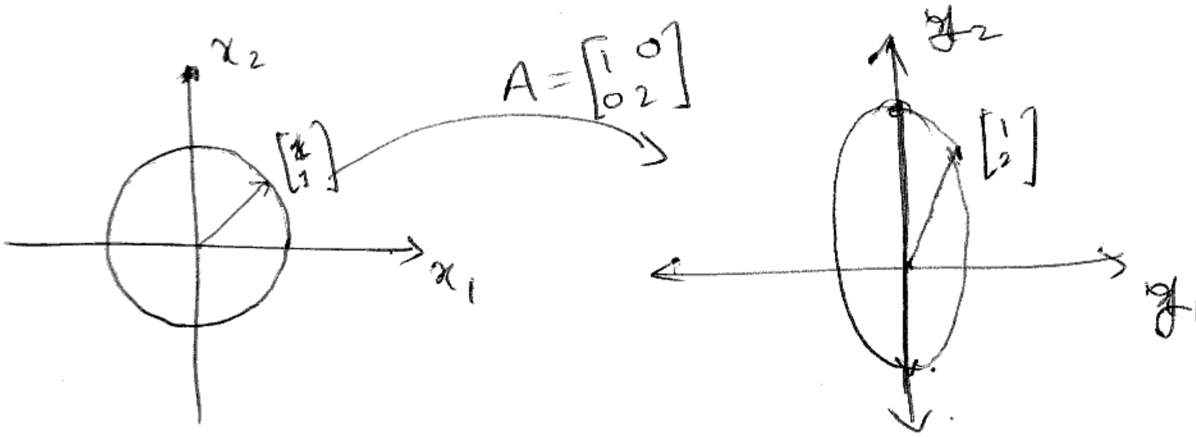
Thus

$$\|A\|_2 = \max_{\|x\|_2 = 1} \frac{\|Ax\|_2}{\|x\|_2}.$$

~~Two~~ Picture in $\mathbb{R}^2$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$$

Consider $\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1^2 + x_2^2 = 1 \right\}$.



$$\frac{\|Ax\|}{\|x\|} = \frac{y_1^2 + y_2^2}{x_1^2 + x_2^2}$$

$$= y_1^2 + y_2^2$$

$$y = Ax = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}.$$

$$x_1 = y_1 \qquad x_2 = \frac{y_2}{2}$$

$$y_1^2 + \frac{y_2^2}{4} = 1$$

$$\max \left(y_1^2 + y_2^2 \right) = 4 \quad \text{for } x = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$\|A\|_2 =$ maximum magnification that one can obtain and direction of maximum magnification is x for which $\max_{\|x\|_2 = 1} \|Ax\|_2$ occurs.

$\boxed{\text{One - norm.}}$ is max{column sums of A}

$$\|A\|_1 = \max_{\|x\|_1 = 1} \|Ax\|_1 .$$

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\cancel{\text{then}}\ Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{bmatrix}$$

$$\|Ax\|_1 = |a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n|$$
$$+ |a_{21}x_1 + \cdots + a_{2n}x_n|$$
$$+ \quad \vdots$$
$$+ |a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n|$$

$$\cancel{\leq |a_{11}||x_1| + |a_{12}||x_2| + \cdots + |a_{1n}||x_n|}$$

we know. $|x_1| + |x_2| + \cdots + |x_n| = 1$

$$\|Ax\|_1 \leq \left[|a_{11}|\,|x_1| + \cdots + |a_{1n}|\,x_n\right]$$
$$+ \left[|a_{21}|\,|x_1| + \cdots + |a_{2n}|\,x_n\right]$$
$$+ \cdots + \left[|a_{n1}x_1| + |a_{n2}|\,|x_2| + \cdots + |a_{2n}|\,|x_n|\right]$$

$$= |x_1|\left(|a_{11}| + \cdots + |a_{n1}|\right)$$
$$+ |x_2|\left(|a_{12}| + |a_{22}| + \cdots + |a_{n2}|\right)$$
$$+ \vdots$$
$$+ |x_n|\left(|a_{n1}| + \cdots + |a_{nn}|\right)$$

$$= (\text{colsum } 1)\,|x_1| + \cdots + (\text{colsum } n)\,|x_n|$$

$$\text{maxcolsum} = \max\{\text{colsum } 1, \ldots, \text{columnsum } n\}$$

then $\|Ax\|_1 \leq \text{maxcolsum}\left(|x_1| + \cdots + |x_n|\right) = 1$

$\leq$ max colsum.

When is $\|Ax\|_1 = $ max colsum? for

$$x = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$ ← index of column for which sum is maximum

then $\max \|Ax\|_1 = $ maxcolsum.

Similarly prove for $\|A\|_\infty = \max\left\{\begin{array}{c}\text{raw sums} \\ \text{of } A.\end{array}\right\}$