LQR - problem. is an Optimal Control Problem.
(Linear Quadratic Regulator)

- Optimal Control Problem (Typical form)

minimize Cost "functional"

$$J = \underbrace{\int_0^{t_f} g(x(t), u(t), t)\, dt}_{\text{Cost accumulated by running system in interval } [0, t_f]} + \underbrace{h(x(t_f), t_f)}_{\text{Cost incurred at the terminal time.}}$$

Subject to system differential equation.

$$\dot{x}(t) = f(x(t), u(t), t)$$

$$x(0) = x_0 \text{, free}$$

$$t_f \text{ fixed.}$$

- Objective is to choose input $u(t)$ s.t J is minimized.

- Soft Constraints can be imposed on Control System operation using cost functional of type described above

- LQR specifies a cost functions of the form.

$$J = \int_0^{t_f} \left[x(t)^T Q\, x(t) + u(t)^T R\, u(t) \right] dt + x(t_f)^T N\, x(t_f)$$

where $Q, R, N > 0$ are positive definite matrices

- One can tune Q, R, N to choose Controller $u(t)$ for specific speed of convergence to set points, etc..

One can solve the optimal control problem using Pontryagin's Minimum Principle. Let us see what it is.

Thm: [Pontryagin's Minimum Principle].

~~The~~ For the optimal control $u^*(t)$ & optimal state trajectory $x^*(t)$, there exists non-zero $p(t)$ s.t following equations are satisfied.

(i) $$\frac{dp}{dt} = -\frac{\partial H}{\partial x}^T$$

(ii) $$\dot{x} = \frac{\partial H}{\partial p}.$$

(iii) $$p^T(t_f) = \left.\frac{\partial h}{\partial x}\right|_{t=t_f}$$

and $$u^*(t) = \arg\min_u H(x, p, u, t).$$

where $$H(x,p,u,t) \cancel{= x^TQx + u^TRu + p^T(f(x,u,t)}$$
$$= g(x,u,t) + p^T f(x,u,t).$$

We apply it to LQR problem. (finite horizon)

$$\min \quad J = \int_0^{t_f} \frac{1}{2}(x^T Q x + u^T R u)\,dt + \frac{1}{2} x^T(t_f) N x(t_f)$$

$$\text{s.t.} \quad \dot{x} = Ax + Bu$$

x_0 specified.

t_f fixed.

Apply necessary conditions :

$$H(x,u,p,t) = \frac{1}{2}(x^T Q x + u^T R u) + p^T(Ax + Bu)$$

(i) $\dot{p} = -\dfrac{\partial H}{\partial p}^T = -A^T p - Qx$

(ii) $\dot{x} = Ax + Bu.$

(iii) $p(t_f) = \dfrac{\partial h}{\partial x(t_f)} = N x(t_f)$

To find

$$u^* = \arg\min_u H(x,u,p,t)$$

use

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u^T R + p^T B = 0.$$

$$\Rightarrow u = -R^{-1} B^T p$$

Although $x(t_f)$ is free

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \quad x(0) = x_0, \; p(t_f) = N x(t_f)$$

Two point Boundary Value problem

from (iii) we guess that nature of

$p(t) = K(t)\, x(t)$ where $K(t)$ is matrix function. of time.

and at $t = t_f$, $K(t_f) = N$.

Thus $u(t) = -R^{-1} B^T K(t)\, x(t)$

also

$$\dot{p}(t) = \dot{K}(t)\, x(t) + K(t)\, \dot{x}(t)$$

$$= \dot{K} x + K(Ax + Bu)$$

$$-Qx - A^T p = \dot{K} x + KAx + KBu$$

$$-Qx - A^T K x = \dot{K} x + KAx - KBR^{-1}B^T K x$$

$$\Rightarrow \quad \dot{K} + KA + A^T K + Q - KBR^{-1}B^T K = 0$$

since $x(t) \neq 0$.

$$\dot{K} + KA + A^T K + Q - KBR^{-1}B^T K = 0 \quad -(1)$$

(Ricatti Differential equation)

with $K(t_f) = N$.

Thus to solve finite horizon LQR problem

One gets a time varying state-feedback gain matrix using $K(t)$ as follows

Solve (1) for $K(t)$

use $u^*(t) = -R^{-1} B^T K(t) x(t)$

as optimal Controller.

- Infinite Horizon Case.

~~K(t) As we solve vary as~~ : As we ~~start~~ move backwards from t_f, $K(t_0)$ quickly converges to equilibrium [See Donald Kirk or any other Optimal contol book]

Roughly arguing (Because of above) in infinite Horizon case we need solution to Algebraic Riccatti equation ~~to~~ which is equilibrium solution to Differential Riccatti eqn. (1).

$$A^T K + KA + Q - KBR^{-1}B^T K = 0$$

ex: $\dot{x} = x + u$. find u ~~to~~

$$\min J = \int_0^1 \tfrac{1}{2}(x^2 + u^2)\, dt + \tfrac{1}{2} x(1)^2$$

$Q = 1, \quad R = 1, \quad N = 1$

$$\underbrace{\left.\begin{aligned} \dot{K} &= +2K - K^2 + 1 \\ K(1) &= 1 \end{aligned}\right]}_{\text{~~Problem~~}}$$

Solve it $\displaystyle\int \frac{dK}{2K - K^2 + 1} = \int dt = t + c.$

find c using $K(1) = 1$.

Controllability Gramian $\quad J = \int_0^\infty u^T u \, dt$

ARE $\quad A^T K + KA - KBR^{-1}B^T K = 0.$

put $P = K^{-1}$ and multiply on both sides (we seek symmetric K)

$$\underline{PA^T + AP = BB^T} \quad \text{(Lyapunov eqn with } Q = BB^T)$$

Solution to this eq. is Controllability Gramian.

$$P = \int_0^\infty e^{A^T t} BB^T e^{At} \, dt$$