Lecture 5.

Sensitivity of $Ax = b$

Consider $Ax = b$.

A - non-singular. $\quad$ $b \in$ span of columns of A

$b \neq 0$.

then unique $x \neq 0$ s.t $Ax = b$.

perturb b by δb and consider

new system.

$$A\hat{x} = b + \delta b.$$

Will it also have a unique $\hat{x}$.

Yes.

How far is x from $\hat{x}$? Let $\delta x = \hat{x} - x$.

Can we say that for small δb, δx will be small?

Relative error?

But we have vectors here!

Why did we introduce norms earlier!

Precisely for this reason.

Size of δb relative to b is given by

$$\frac{\|\delta b\|}{\|b\|} \qquad \text{similarly} \qquad \frac{\|\delta x\|}{\|x\|}$$

Now for small $\frac{\|\delta b\|}{\|b\|}$

is $\frac{\|\delta x\|}{\|x\|}$ small?

Lets find out!

$$Ax = b.$$

$$A(x+\delta x) = b + \delta b.$$

then. $\delta x = A^{-1}\delta b.$

$$\boxed{\|\delta x\| \leq \|A^{-1}\| \, \|\delta b\|} \quad \text{--- } (1)$$

Also $b = Ax \Rightarrow \|b\| \leq \|A\| \|x\|.$

Or
$$\boxed{\frac{1}{\|x\|} \leq \|A\| \frac{1}{\|b\|}} \quad \text{--- } (2).$$

(1) & (2) gives

$$\boxed{\frac{\|\delta x\|}{\|x\|} \leq \|A\| \, \|A^{-1}\| \frac{\|\delta b\|}{\|b\|}} \quad \text{--- } (3)$$

Condition number.

$$\kappa(A) := \|A\| \, \|A^{-1}\|$$

1 & 2 are Sharp inequalities

i.e. equality is possible.

3 is also sharp.

Easy to see:

(i) $K(A) = K(A^{-1})$

(ii) $K(cA) = K(A)$ for any $c \neq 0$ (scalar)

What does 3 tell us about sensitivity?

It says that if $K(A)$ is not too big small changes in b gives small changes in x.

On the other hand if $K(A)$ is large there is no such guarantee.

Due to sharpness of 3 there is δb & b for which $\frac{\|\delta x\|}{\|x\|}$ is largers. ~~there~~

$\boxed{\text{ill Conditioned system!}}$

Some more observations about

$\kappa(A) \geq 1$. (why?)

$$1 = \|I\| = \|AA^{-1}\| \leq \|A\| \, \|A^{-1}\| = \kappa(A)$$

So best possible value is 1 !

What is worst value ? Very large no.

When A is close to being singular.

$$\kappa_p(A) = \|A\|_p \, \|A^{-1}\|_p$$

for $1 \leq p \leq \infty$

1-norm, ∞-norm, 2-norm.

Ex: $A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}$

Calculate

$\|A\|_\infty = 1999 = \|A\|_1$

$\|A^{-1}\|_\infty = \|A^{-1}\|_1 = 1999$

$A^{-1} = \begin{bmatrix} -998 & 999 \\ 999 & -1000 \end{bmatrix}$

$\kappa_1(A) = \kappa_\infty(A) = (1999)^2$

which is very high.

Recall geometric picture we saw earlier.

$$\begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$$1000 x_1 + 999 x_2 = b_1$$
$$999 x_1 + 998 x_2 = b_2.$$

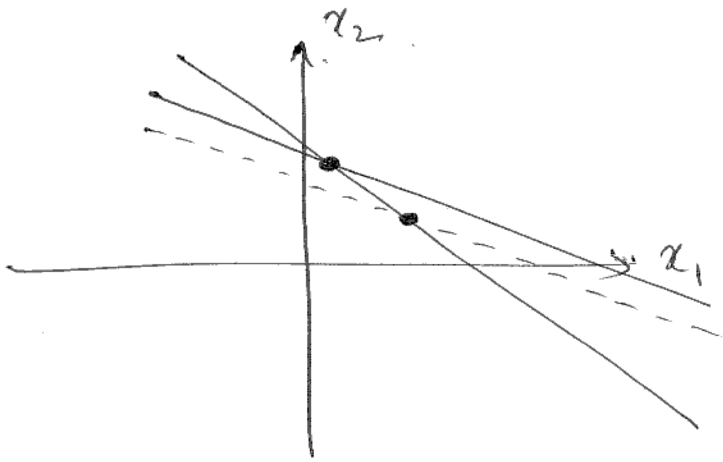
$$x_2 = \frac{b_1}{999} + \overset{m_1}{\frac{1000}{999}} x_1$$

$$x_2 = \frac{b_2}{998} + \frac{999}{998} x_1$$

Calculate Slopes. $m_1 = \frac{-1000}{999}$

& $m_2 = \frac{-999}{998}$

Nearly parallel lines.



This diagram is for demonstration. Actual ill-conditioning leads to lines even more parallel.

In practices we do not know b, we use $\underline{b+\delta b}$ & solve the system. $A\hat{x} = \underline{b+\delta b}$

Let say $\frac{\|\delta b\|}{\|b\|} \approx 10^{-4}$

was specifed. (Tolerance)

Now solving we get $x + \delta x$

and not x.

then $\frac{\|\delta x\|}{\|x\|} \lesssim K(A)\, 10^{-4}$

Now if $K(A) \leq 10^{2}$

then in worst case $\frac{\|\delta x\|}{\|x\|} \lesssim 10^{-2}$

which is in most cases

acceptable.

but if lets say $K(A) \approx 10^{4}$

then it is possible that

$\frac{\|\delta x\|}{\|x\|} \approx 1$

which is unacceptable.

It seems that range 10^{2} to 10^{4} is sort of a boundary betn well-conditioned & ill-conditioned.

It can also be decided by floating pt. arithmetic.

· b may be accurate but while storing on computer we may loose some decimal places & be forced to work with

$b + \delta b$.

Then again $K(A)$ will depend on how accurately we are able to store b well in computer.

if we are able to store up to 6 decimal places. then $\frac{\|\delta b\|}{\|b\|} \approx 10^{-7}$

Then ~~even~~ $K(A) \approx 10^{7}$ would be problematic

But, $K(A)$ of 10^{3}, 10^{4}, may be good enough for our purposes.

Maximum & minimum magnification

$$\max mag(A) = \max_{\|x\|=1} \|Ax\|$$

$$\min mag(A) = \min_{\|x\|=1} \|Ax\|$$

(i) $\max mag(A) = \dfrac{1}{\min mag(A^{-1})}$

$$\max_{\|x\|=1} \|Ax\| = \frac{1}{\min_{\|x\|=1} \|A^{-1}x\|}$$

$$\operatorname{maxmag}(A) = \max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}$$

Let $Ax = y$

$$\operatorname{maxmag}(A) = \max_{\|x\| \neq 0} \frac{\|y\|}{\|x\|}$$

$$= \max_{\|x\| \neq 0} \frac{\|y\|}{\|A^{-1}y\|}$$

$$= \max_{\|y\| \neq 0} \left[\frac{1}{\frac{\|A^{-1}y\|}{\|y\|}} \right]$$

$$= \frac{1}{\min_{\|y\| \neq 0} \left[\frac{\|A^{-1}y\|}{\|y\|} \right]}$$

$$= \frac{1}{\operatorname{minmag}(A^{-1})}$$

Similarly.

$$\operatorname{minmag}(A) = \frac{1}{\operatorname{maxmag}(A^{-1})}$$

then $$K(A) = \|A\| \, \|A^{-1}\| = \frac{\operatorname{maxmag}(A)}{\operatorname{minmag}(A)}$$

So. if minmag(A) << maxmag(A),
~~re~~sults in
illconditioning.

for Singular A

minmag(A) = 0 ?

Also, det(A) = 0.
Is determinant related to the condition.
- no. ?

Ex:

$$A = \begin{bmatrix} \cancel{90} & 0 \\ 0 & 90 \end{bmatrix}$$

det(A) = $(90)^2$ = 8100
but $\kappa(A) = 1$

Scaling affects determinant but not condition number.

Ex: From Watkin's book

$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}$$

$\|A\|_\infty = 1999$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. gives maxmag. direction

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -998 & 999 \\ 999 & -1000 \end{bmatrix}$$

$\|$~~A~~$\|$

$$A^{-1} \begin{bmatrix} \cancel{1997} \; -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1997 \\ -1999 \end{bmatrix}$$

$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -998 & 999 \\ 999 & -1000 \end{bmatrix}$$

max mag(A)

$= \|A\|_\infty = 1999$

minmag(A)

$$= \frac{1}{\text{max mag}(A^{-1})}$$

$$= \frac{1}{\|A\|_\infty} = \frac{1}{1999}.$$

direction of maxmag.

$\arg\max_{\|x\|_\infty = 1} \|Ax\|_\infty$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

since $A\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$ lies in the

direction of minmag

of A^{-1}

direction of ~~max~~ maxmag.

of A^{-1}

$$x = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

since $A^{-1}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1997 \\ -1999 \end{bmatrix}$

lies in the

direction of minmag

of A

$$\begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}.$$

$$Ax = b$$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

and

$$\begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1998.99 \\ 1997.01 \end{bmatrix}.$$

$$A\hat{x} = b + \delta b \quad , \quad \delta b = \begin{bmatrix} -0.01 \\ 0.01 \end{bmatrix} = 0.01 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

δb is in the direction of max. mag. of A^{-1}.

$$\hat{x} = x + \delta x.$$

$$\delta x = \begin{bmatrix} 19.97 \\ -19.99 \end{bmatrix}.$$

$$\hat{x} = \begin{bmatrix} 20.97 \\ -18.99 \end{bmatrix}.$$

if system is ill conditioned we can build such examples.

$$\boxed{\frac{\|\delta b\|}{\|b\|} \leq K(A) \frac{\|\delta x\|}{\|x\|}} \quad — \text{IE}$$

Example we saw was

small change in b
gives
large change in x

IE → small change in x may occur ~~for~~
even if we change b significantly

For such system solving for
x is meaningless!
The soln may jump places ~~even~~ in
presence of uncertainties!

$$\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$$

$$K_1(A) = K_2(A) = K_\infty(A) = \frac{1}{\epsilon}$$

$A = [a_1 \; a_2 \; \cdots \; a_n]$ then.

then $\kappa_p(A) \geq \dfrac{\|a_i\|_p}{\|a_j\|_p}$, $1 \leq p \leq \infty$.

~~as~~ Clearly $Ae_i = a_i$

$Ae_j = a_j$

$$\text{maxmag}(A) = \max_{\|x\| \neq 0} \frac{\|Ax\|_p}{\|x\|_p} \geq \frac{\|Ae_i\|_p}{\|e_i\|_p} = \|a_i\|_p$$

Similarly

$\text{minmag}(A) \leq \|a_j\|_p$.

$$\kappa_p(A) = \frac{\text{maxmag}(A)}{\text{minmag}(A)} \geq \frac{\|a_i\|_p}{\|a_j\|_p}.$$

So scaling is critical

if one column differs from other

by orders of magnitude.

- Estimating condition no.

cond(A) - Matlab.

A large then calculating A^{-1} is huge ask.

In general no need of perfection. Order of magnitude is enough.

Estimate of $\kappa(A)$

$$\kappa_1(A) = \|A\|_1 \|A^{-1}\|_1$$

$\|A\|_1$ easy. column sum.

we need estimate $\|A^{-1}\|_1$

for any $b \neq 0 \in \mathbb{R}^n$.

$$\frac{\|A^{-1}b\|_1}{\|b\|_1} \leq \max_{y \neq 0} \frac{\|A^{-1}y\|_1}{\|y\|_1} = \|A^{-1}\|_1$$

let ~~...~~ , $A^{-1}b = x$

$$\frac{\|x\|_1}{\|b\|_1} \leq \|A^{-1}\|_1$$

and $\kappa_1(A) \geq \|A\|_1 \frac{\|x\|_1}{\|b\|_1}$

we get a lowerbound.

1. LU decomposition of A.
2. Calculate x from it $Ax = b = w$.

roughly. $O(n^2)$

if b is chosen in direction near max mag of A^{-1} then estimate

$$\kappa_1(A) \approx \|A\|_1 \frac{\|A^{-1}b\|_1}{\|b\|_1}$$

In general take different directions b and compute $\kappa_1(A)$.

Perturbation in A (the coefficient matrix

$$Ax = b.$$

$$(A + \delta A)\hat{x} = b.$$

Assumption: $\left[\frac{\|\delta A\|}{\|A\|} \text{ is small} \right]$

Now A is non-singular.

- x is unique soln to $Ax = b$.
- Same cannot be said about
  $(A + \delta A)\hat{x} = b$.
- Can we ensure this!
  i.e., When is $A + \delta A$ non-singular.

Thm: If A is non-singular and $\frac{\|\delta A\|}{\|A\|} < \frac{1}{\kappa(A)}$ then $(A+\delta A)$ is non-singular

Proof [If $A + \delta A$ is singular.

then $\|\delta A\| \, \|A^{-1}\| \geq 1$]?

if $A + \delta A$ is singular.
there is $y \neq 0 \in R^n$
$(A + \delta A)y = 0.$

$Ay = -\delta A y$

$y = -A^{-1} \delta A y$

$\cancel{\|y\|} \leq \|A^{-1}\| \, \|\delta A\| \, \cancel{\|y\|}$

$\|A^{-1}\| \, \|\delta A\| \geq 1.$

If $\|\delta A\| \, \|A^{-1}\| < 1$ then.

$A + \delta A$ is non-singular.

i.e., if
$$\frac{\|\delta A\|}{\|A\|} < \frac{1}{\|A\| \, \|A^{-1}\|} < \frac{1}{k(A)}$$

then $A + \delta A$ is non-singular.

* Distance ~~to~~ ~~so~~ of a matrix to non-singularity.

Given A - non-singular.

$A + \delta A$ singular $\Rightarrow$ $\dfrac{\|\delta A\|}{\|A\|}$ is atleast $\dfrac{1}{K(A)}$.

[Relative error in A].

for 2-norm. (spectral norm)

$$\boxed{\frac{\|\delta A\|_2}{\|A\|_2} = \frac{1}{K_2(A)}}$$

Proof later!