Let us look at what happens to the soln. if we perturb coefficient matrix A by δA s.t. $(A+\delta A)$ remains non-singular

If A was invertible then.
$A+\delta A$ will remain singular if

if $\dfrac{\|\delta A\|}{\|A\|} < \dfrac{1}{k(A)}$. (as we saw in previous lecture)

Now, $Ax = b$ (x is a solution to $Ax = b$)

Let $\hat{x} = x + \delta x$ be soln to the perturbed system of eqn.

$$(A+\delta A)(x+\delta x) = b.$$

Then $Ax + A\,\delta x + \delta A\, x + \delta A\,\delta x = b$

but $Ax = b$

Thus, $A\,\delta x + \delta A\, x + \delta A\,\delta x = 0$

$$\delta x = -A^{-1}\delta A x - A^{-1}\delta A\,\delta x$$

$$\Rightarrow \|\delta x\| \leq +\|A^{-1}\|\,\|\delta A\|\,\|x\| + \|A^{-1}\|\,\|\delta A\|\,\|\delta x\|$$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \leq \|A^{-1}\|\,\|A\|\,\frac{\|\delta A\|}{\|A\|} + \|A^{-1}\|\,\|A\|\,\frac{\|\delta A\|}{\|A\|}\,\frac{\|\delta x\|}{\|x\|}$$

$$\Rightarrow \left(1 + k(A)\frac{\|\delta A\|}{\|A\|}\right)\frac{\|\delta x\|}{\|x\|} \leq k(A)\frac{\|\delta A\|}{\|A\|}$$

$$\Rightarrow \boxed{\frac{\|\delta x\|}{\|x\|} \le \frac{k(A)\,\dfrac{\|\delta A\|}{\|A\|}}{1 - k(A)\,\dfrac{\|\delta A\|}{\|A\|}}} \quad (*)$$

This gives us relation between relative error in x, condition number and relative error in A.

It says that, in terms of relative error in A i.e., $\frac{\|\delta A\|}{\|A\|}$ and $k(A)$,

relative error in solution is upper bounded by $(*)$

If A was well-conditioned and $\frac{\|\delta A\|}{\|A\|}$ was small, then

$$k(A)\,\frac{\|\delta A\|}{\|A\|} \ll 1$$

and we roughly recover a direct relation

$$\boxed{\frac{\|\delta x\|}{\|x\|} \lesssim k(A)\,\frac{\|\delta A\|}{\|A\|}}$$

which is same as for case in which b was perturbed.

This is rough inequality! Not exact!

Now consider case where both

A and b are perturbed by δA & δb respectively.

Let A be invertible.

Again, we consider δA s.t $A+\delta A$

remains invertible

Let x be soln of $Ax = b \Rightarrow \|b\| \le \|A\|\|x\|$

$\Rightarrow \frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|}$

then $x = A^{-1}b \Rightarrow \|x\| \le \|A^{-1}\|\|b\|$

$\Rightarrow \frac{1}{\|b\|} \le \frac{\|A^{-1}\|}{\|x\|}$

Let $\hat{x} = (x+\delta x)$ be soln to perturbed system

Then $(A+\delta A)(x+\delta x) = b+\delta b$

Now $\cancel{A}x + A\delta x + \delta A x + \delta A \delta x = \cancel{b} + \delta b$

$$\delta x = A^{-1}\delta b - A^{-1}\delta A x - A^{-1}\delta A \delta x$$

$$\|\delta x\| \le \|A^{-1}\|\|\delta b\| + \|A^{-1}\|\|\delta A\|\|x\| + \|A^{-1}\|\|\delta A\|\|\delta x\|$$

$$\frac{\|\delta x\|}{\|x\|} \le \frac{\|A\|\|A^{-1}\|\|\delta b\|}{\|A\|\|x\|} + \|A^{-1}\|\|A\|\frac{\|\delta A\|}{\|A\|} + \|A^{-1}\|\|A\|\frac{\|\delta A\|}{\|A\|}\frac{\|\delta x\|}{\|x\|}$$

$$\le k(A)\frac{\|\delta b\|}{\|b\|} + k(A)\frac{\|\delta A\|}{\|A\|} + k(A)\frac{\|\delta A\|}{\|A\|}\frac{\|\delta x\|}{\|x\|}$$

$$\frac{\|\delta x\|}{\|x\|}\left(1 - k(A)\frac{\|\delta A\|}{\|A\|}\right) \le k(A)\left(\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|}\right)$$

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{k(A)\left(\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|}\right)}{\left(1 - k(A)\frac{\|\delta A\|}{\|A\|}\right)}$$

This is the bound on relative error in x when both A and b are perturbed by δA & δb respectively.