# State observer

$$u \longrightarrow \boxed{\begin{array}{c} \dot{x} = Ax + Bu \\ y = Cx \end{array}} \longrightarrow y$$

- $x_0$ is not known
- $y_0$, $y(t)$, $u(t)$ at each time $t$ is known.
- Can we compute $x_0$ from what is known?

One can try to compute derivatives of output to get more information about state trajectory.

$$y = Cx$$

$$y^{(1)} = C\dot{x} = CAx + CBu$$

$$y^{(2)} = C\ddot{x} = CA\dot{x} + CBu^{(1)}$$
$$= CA^2x + CABu + CBu^{(1)}$$

$$\vdots$$

$$y^{(n-1)} = CA^{n-1}x + CA^{n-2}Bu + CA^{n-3}Bu^{(1)}$$
$$+ \cdots + CBu^{(n-2)}$$

Let $\mathcal{Y} = \begin{bmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n-1)} \end{bmatrix}$ $\quad \mathcal{U} = \begin{bmatrix} u \\ u^{(1)} \\ u^{(2)} \\ \vdots \\ u^{(n-1)} \end{bmatrix}$

$$\begin{bmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\mathcal{O}_{(C,A)}} x(t) + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ CB & 0 & \cdots & 0 \\ CAB & CB & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-2}B & CA^{n-3}B & \cdots\ CB & 0 \end{bmatrix}}_{\Gamma_{(C,A,B)}} \begin{bmatrix} u \\ u^{(1)} \\ u^{(2)} \\ \vdots \\ u^{(n-2)} \\ u^{(n-1)} \end{bmatrix}$$

$$\mathcal{Y} = \mathcal{O}_{(C,A)} x(t) + \Gamma_{(C,A,B)} \mathcal{U}$$

If $\mathcal{O}_{(C,A)}$ is invertible ~~then~~

$$\Downarrow$$

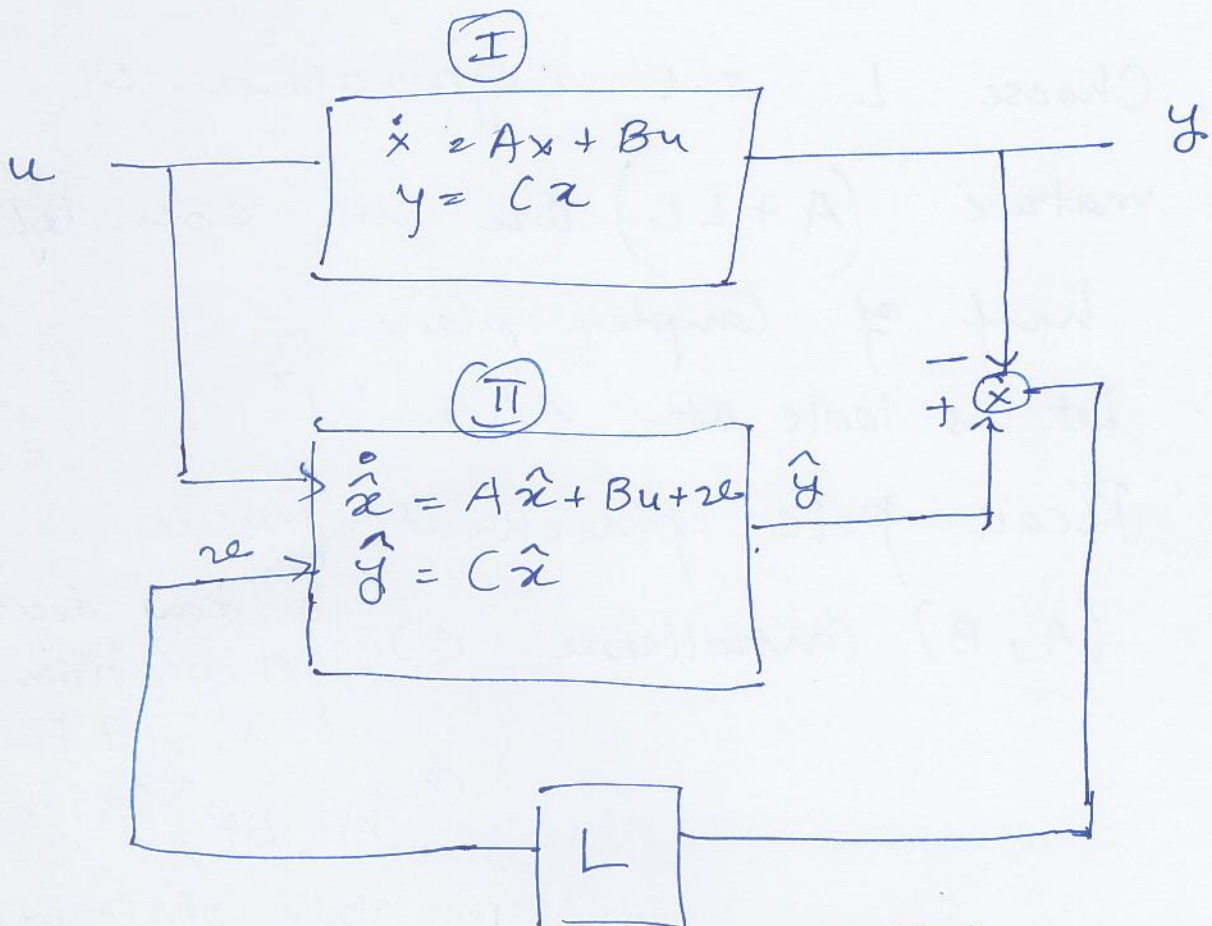
$$x(t) = \mathcal{O}^{-1}_{(C,A)} \left[ \mathcal{Y} - \Gamma_{(C,A,B)} \mathcal{U} \right]$$

i.e State trajectory can be reconstructed from output

But, typically $y(t)$, $u(t)$ contains noise and taking derivatives leads to problems of noise getting amplified.

Is there any means by which taking derivatives can be avoided?

Yes one can construct full state observer as follows.




Block (I) is system. Block (II) is on computer.

Choose $v = L(\hat{y} - y)$ in Block II

Let $\hat{x} - x = e$ be error between States of two blocks

Then $\dot{e} = \dot{\hat{x}} - \dot{x}$

$$= A\hat{x} + \cancel{Bu} + v - Ax - \cancel{Bu}$$

$$= A(\hat{x} - x) + L(\hat{y} - y)$$

$$= A(\hat{x} - x) + L(C\hat{x} - Cx)$$

$$= (A + LC)(\hat{x} - x) = (A + LC)e$$

Choose $L$ s.t eigenvalues of matrix $(A+LC)$ are in open left half of Complex plane.

Let us look at $A^T + C^T L^T$

Recall pole placement problem.

$(A, B)$ controllable $\Leftrightarrow$ for any degree $n$ polynomial $p(\lambda)$, $\exists F$ s.t $\chi_{(A+BF)}(\lambda) = p(\lambda)$

i.e pole placement possible

Thm: $(C, A)$ observable

$$\Updownarrow$$

$(A^T, C^T)$ Controllable

$$\Updownarrow$$

$\forall\ p(\lambda) \in \mathbb{R}[\lambda]$ of degree $n$

$\exists\ L^T$ s.t $\chi_{(A^T + C^T L^T)}(\lambda) = p(\lambda)$

$\left[\text{i.e } \chi_{(A+LC)}(\lambda) = p(\lambda)\right]$

Defn: [Detectability] (compare with stabilizability)

Pair $(C, A)$ is detectable if $\exists\ L$

s.t. $\Lambda(A + LC) \subset \mathbb{C}^-$

Thm: $(C, A)$ is detectable if

all $\lambda \in \Lambda(A)$ s.t. $\text{Rank}\begin{bmatrix} C \\ \lambda I - A \end{bmatrix} < n$

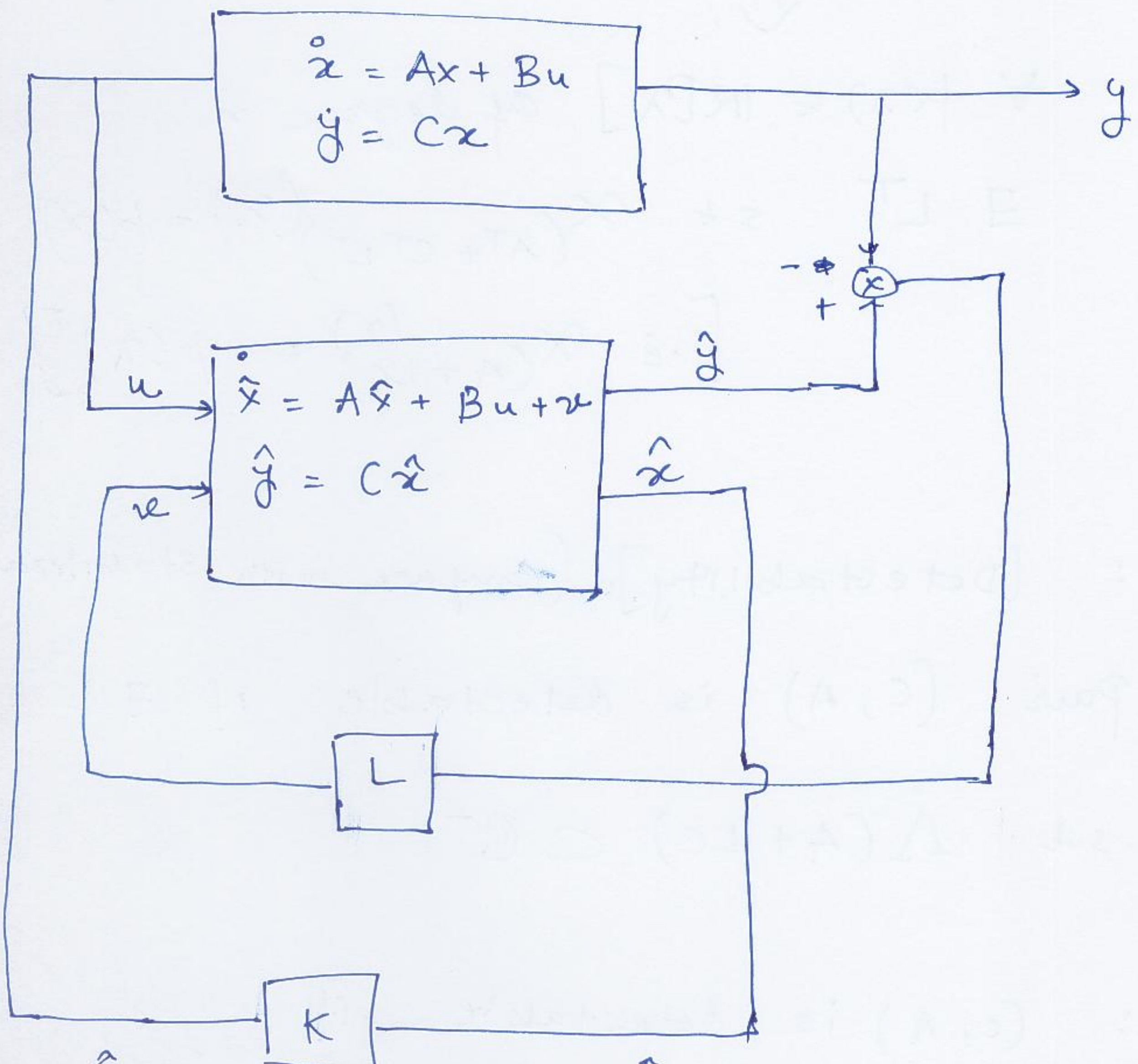
are s.t. $\lambda \in \mathbb{C}^-$

[i.e., in open left half of complex plane].

## Separation principle

Can we use observer designed to stabilize linear system using state feedback?

Yes.




[OVERALL SYSTEM]

Does designing L and K separately s.t $\Lambda(A+LC) \subset \mathbb{C}^-$ and $\Lambda(A+BK) \subset \mathbb{C}^-$ lead to stability of overall system?

$$\dot{x} = Ax + B(K\hat{x})$$

$$\dot{\hat{x}} = A\hat{x} + BK\hat{x} + LC(\hat{x} - x)$$

$$= (A + BK)\hat{x} + LC(\hat{x} - x)$$

$$= (A + BK + LC)\hat{x} - LCx$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Let $$\begin{bmatrix} x \\ e \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}}_{T} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$$T \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

gives us.

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BK & BK. \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

Since K, and L are selected s.t $\Lambda(A + BK) \subset \mathbb{C}^-$, $\Lambda(A + LC) \subset \mathbb{C}^-$, we get overall system to be stable.