# Practice Problems

Linear algebra.

(1) Consider $P_3 := \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$.

be a set of polynomials of degree $\leq 3$

A particular basis for this space is

$B_s = \{1, x, x^2, x^3\}$. A derivative map $\frac{d}{dx} : P_3 \to P_3$ is a linear map and can be expressed in $B_s$ as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $B_c = \{1, x, 2x^2 - 1, 4x^3 - 3x\}$

be a new basis. Express $\frac{d}{dx}$ as a linear map in this new basis.

Q.2 Consider a ~~matrix~~ linear map $A : \mathbb{R}^5 \to \mathbb{R}^5$. Let $b \in \mathbb{R}^5$ be a column vector. ~~It is~~ suppose it is known that

$$A^5 + A^4 + A^3 + A^2 + A + I = 0.$$

Also the set $\{b, Ab, A^2b, \ldots, A^4b\}$ contains linearly independent vectors. Find matrix $A$ by selecting a suitable basis.

Q.3: Given

$$A = \left[\begin{array}{ccc|cc} 1 & 101 & \pi & 1 & \pi \\ 1001 & 999 & e & e^2 & \pi^{1.5} \\ 159 & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & e^3 & e^\pi \\ \hline 0 & 0 & 0 & -1 & \pi^e \\ 0 & 0 & 0 & \sqrt{13} & \sqrt{3} \end{array}\right]$$

Find an $A$-invariant subspace. Write basis for it.

Q.4 Let $A \in \mathbb{R}^{n \times n}$ be a matrix. Show that if $\lambda$ is an eigenvalue of matrix $A$, then for any polynomial $p(x)$, $p(\lambda)$ is an eigenvalue of $p(A)$. Moreover if $v$ is eigenvector of $A$ corresponding to eig. val. $\lambda$, then

$v$ is also an eigenvector of $p(A)$ corresponding to eigenvalue $p(\lambda)$.

Q.5 Defⁿ 1: [Monic Polynomial]

A polynomial $p(\lambda) \in \mathbb{R}[\lambda]$ is called monic if its leading coefficient is 1.

Defⁿ 2: [Minimal polynomial of A, (m.p. of A)]

The least degree monic polynomial $p(\lambda)$ s.t. $p(A) = 0$ is called as the "minimal polynomial" of A.

Note characteristic polynomial of A is $\pi(\lambda) = \det(\lambda I - A)$.

In general, $p(\lambda) \neq \pi(\lambda)$.

(i) Show that $p(\lambda)$ divides $\pi(\lambda)$

(ii) Show that $\deg \pi(\lambda) = n$, $\deg p(\lambda) \leq \deg \pi(\lambda)$

(iii) For $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, find the minimal polynomial and characteristic polynomial of A.

Q. 6. Let $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Generate vectors $b, Ab, A^2b, A^3b, A^4b, A^5b$

It is clear ~~Show~~ that

$$\mathrm{Span}\{b\} \subset \mathrm{Span}\{b, Ab\} \subset \mathrm{Span}\{b, A^2b, A^2b\}$$
$$\subset \cdots \subset \mathrm{Span}\{b, Ab, A^2b, A^3b, \cdots A^5b\}$$

Find ~~minimum number~~ minimum $k$ s.t

$$\mathrm{Span}\{b, Ab, \cdots, A^kb) = \cancel{\mathrm{Span}\{b, Ab, \cdots, A^kb, A^{k+1}b\}}$$
$$= \mathrm{Span}\{b, Ab, A^2b, \cdots, A^{k+1}b\}.$$

~~Let k be the min~~

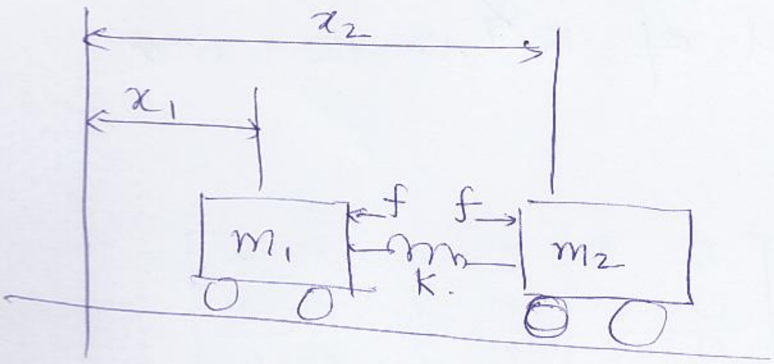
Show that this implies that

$b, Ab, A^2b, \cdots, A^{k+1}b$ are linearly dependent.

Find linear dependence relation among them.

# Reachable space

Q. 1




$$m_1 \ddot{x}_1 + K(x_1 - x_2) + f = 0$$
$$m_2 \ddot{x}_2 + K(x_2 - x_1) + f = 0$$

Example from Control system design B. Friedland Chapter 5
Many more such examples!

$$x = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}, \quad u = f$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/m_1 & 0 & K/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/m_2 & 0 & -K/m_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/m_1 \\ 0 \\ -1/m_2 \end{bmatrix} u$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/m_1 & 0 & K/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/m_2 & 0 & -K/m_2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1/m_1 \\ 0 \\ -1/m_2 \end{bmatrix}$$

(i) Find Reachable Space $\mathcal{W}$

(ii) Check if $\mathcal{W} = \mathbb{R}^n$

(iii) If $\mathcal{W} \subsetneq \mathbb{R}^n$ (i.e Strict subset) then use Similarity Transform. decompose $(A, B)$ into controllable & uncontrollable part parts.

(iv) Explain physically why Reachable space is not all of $\mathbb{R}^4$ in this case

Matrix exponential.

Q.2 Let
$$A = \left[\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Find $e^{At}$.

Q.3 Let
$$A = \left[\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & A_{22} \end{array}\right]$$

Show that
$$e^{At} = \left[\begin{array}{c|c} e^{A_{11}t} & 0 \\ \hline 0 & e^{A_{22}t} \end{array}\right]$$

Q.4 Let $A, B$ be such that $AB = BA$.

Show that $e^{A+B} = e^A e^B$

Q.5
$$A = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 3 & 2 & 0 & 1 \\ \hline 0 & 0 & 2 & -3 \\ 0 & 0 & 3 & 2 \end{array}\right],$$
Compute $e^{At}$.