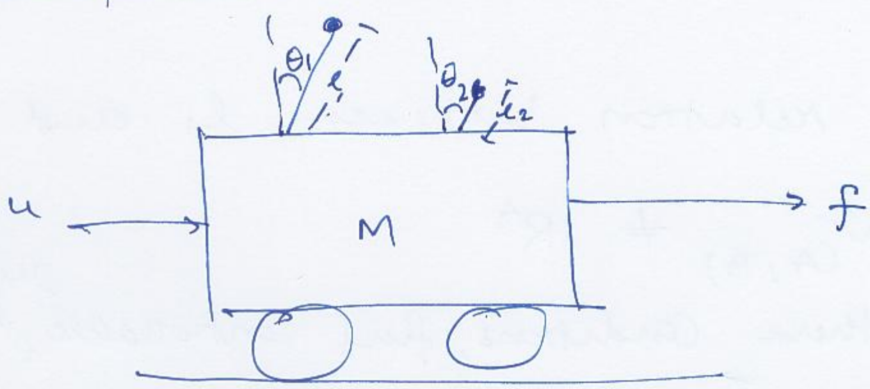
Practice Problems 3

(I)



Two pendulums on cart of lengths l_1 and l_2 with equal mass m. Mass of Cart M

Linearized equations of motion of pendulums are given by.

$$\dot{x} = Ax + Bu, \quad y = Cx$$

$$x = [\theta_1 \;\; \dot{\theta}_1 \;\; \theta_2 \;\; \dot{\theta}_2]$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{g}{l_1} + \frac{mg}{Ml_1} & \frac{mg}{Ml_1} & 0 & 0 \\ \frac{mg}{Ml_2} & \frac{g}{l_2}\left(\frac{M+m}{M}\right) & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{l_1 M} \\ -\frac{1}{l_2 M} \end{bmatrix}$$

$$y = \theta_1, \quad C = [1 \;\; 0 \;\; 0 \;\; 0]$$

~~Assessing~~

Q. 1. Find relation between l_1 and l_2

s.t. $\mathcal{W}_{(A,B)} \neq \mathbb{R}^4$

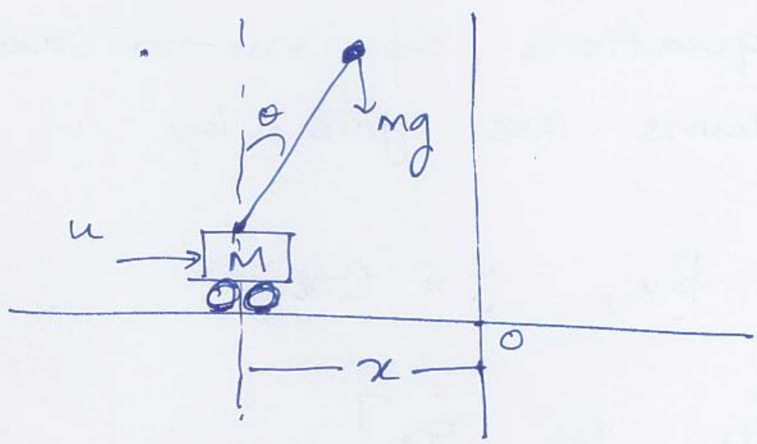
Q. 2 Under these conditions, find controllable uncontrollable decomposition

Q. 3 Find conditions on l_1, l_2 so that

$\operatorname{Ker} \mathcal{O}_{(C,A)} \neq \{0\}$.

Q. 4 Comment on internal stability & external stability of this system?

(II)



Above figure is a schematic for inverted pendulum. Simplified linearized dynamics for small angle θ and small x are as follows

$$\ddot{\theta} = \theta + u, \qquad \ddot{x} = -\left(\frac{3}{4}\frac{m}{M+m}\right)\theta + u$$

(i) Comment on internal stability of this system.

(ii) Comment on external i/p - o/p. stability.

(iii) Suppose $y = x$. Can we estimate θ, $\dot{\theta}$, $\dot{x}$ from y?

(III) Use PBH test to show ~~that~~ the following.

(i) For any $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

there is no $B \in \mathbb{R}^3$ s.t (A, B) is Controllable

(ii) For any $A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

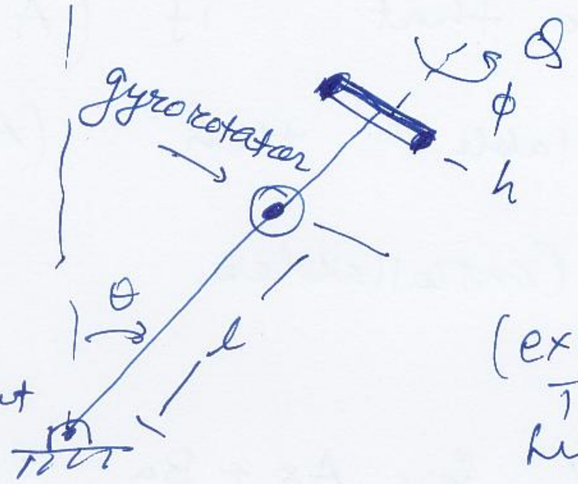
where $\lambda_i \in \mathbb{R}$, $b_i \in \mathbb{R}$

(A, B) Controllable iff $\left[\begin{array}{l} \lambda_i \neq \lambda_j \text{ for } i \neq j \\ \text{and } b_i \neq 0 \text{ for any } i \end{array}\right]$

(IV) Find determinant of

$$V_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

(IV)



$Q \rightarrow$ Control torque

$\theta \rightarrow$ angular displacement of pendulum

(example from T. Kailath Linear Systems)

$\phi \rightarrow$ angular position of gyro-rotator.

Linearized equations of motion are

$$m g l \theta - h \dot{\phi} = I \ddot{\theta} \qquad \Big| \qquad y = \theta$$
$$h \dot{\theta} + Q = J \ddot{\phi}$$

(i) Determine if this system is internally stable. Is it externally stable?

(ii) With Q as the input, is the system controllable? Determine ~~controllable~~ Reachable space.

(iii) With $y = \theta$ as the output, is it observable? Determine Unobservable space.

(iv) Repeat (iii) with $y = \phi$ as the output.

(V) Shoo that if (A, B) is Controllable then $(A + BK, B)$ is also Controllables.

(VI) Suppose $\dot{x} = Ax + Bu$, $y = Cx$ is both observable and controllable. Also let $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times n}$. Then, Shoo that $ABC \neq BCA$