Practice Questions 1

Q.1. For a function $f: \mathbb{R}^n \to \mathbb{R}$ to be a norm, what axioms must it satisfy? List them.

Q.2 Prove that $f: \mathbb{R}^n \to \mathbb{R}$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mapsto \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}$$

is a norm.

Q.3. Prove that $f: \mathbb{R}^n \to \mathbb{R}$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$$

is a norm for $p = 1$. Is it a norm for $p = 0$? Show that as $p \to \infty$, $\left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p} \to \max\{|x_1|, |x_2|, \ldots, |x_n|\}$

Q.4 Draw a set of ~~pts~~ vectors $x \in \mathbb{R}^2$ s.t
a) $\|x\|_2 = 1$
b) $\|x\|_1 = 1$
c) $\|x\|_\infty = 1$

Q.5 Let $B = A^T A$ $(A \in \mathbb{R}^{m \times n})$, show that B is a positive semidefinite matrix

Q.6 Show the Cauchy-Schwarz Inequality using triangle inequality for two norm.

Matrix norms:

Q.1 Show that $\|Ax\| \leq \|A\| \|x\|$ for any norm.

Q.2 Explain what is "induced" about an induced matrix norm?

Q.3 In an expression $\|Ax\|_\alpha \leq \|A\|_\beta \|x\|_\gamma$

What is the difference between $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$ and $\|\cdot\|_\gamma$?

Q.4. Calculate Frobenius norm of

$$\begin{bmatrix} 100 & 1 & -1 \\ -100 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$ Also calculate its 1-norm and ∞-norm.

Q.5 Show that $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$

and $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$

Q.6. Let $v, w \in \mathbb{R}^n$ and $A = vw^T$.

Show that $\|A\|_2 = \|v\|_2 \|w\|_2$

Q.7 Show for $x \in \mathbb{R}^n$.

$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2 \leq n \|x\|_\infty$

Q. 8 use Q. 7 to show.

$$\|A\|_1 \le \sqrt{n}\,\|A\|_2 \le n\,\|A\|_1$$

and $\|A\|_\infty \le \sqrt{n}\,\|A\|_2 \le n\,\|A\|_\infty$

Q. 9 Calculate 2- norm of

(i) $A = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 1 & 100 \\ 100 & 1 \end{bmatrix}$

Also give $x \in \mathbb{R}^2$ s.t. $\|Ax\|_2 = \|A\|_2 \|x\|_2$.

Q. 10 Calculate 1- norm and ∞- norm of matrices in Q. 9. Give $x \in \mathbb{R}^2$ s.t. $\|A\|_1 = \dfrac{\|Ax\|_1}{\|x\|_1}$

Q. 11 Show that for $A \in \mathbb{R}^{n\times n}$, $\|A\|_2 \le \|A\|_F$

Q. 12 ~~Ge~~ Calculate Condition number of matrices in Q. 9. w.r.t 1-norm, 2-norm, ∞- norm. i.e., Calculate $K_1(A)$, $K_2(A)$, $K_\infty(A)$.

Q. 13 Give an example of 4×4 matrix whose Condition number ~~is~~ w.r.t. 1-norm is greater than 1000.

Q. 14 Show that $K_p(A) \ge \max\limits_{1\le i,j\le n} \dfrac{\|a_i\|_p}{\|a_j\|_p}$ where $a_i, a_j \in \mathbb{R}^n$ are columns of matrix A

Q.15 Show that $\kappa_\infty(A) = \kappa_1(A^T)$

and $\kappa_2(A) = \kappa_2(A^T)$

Q.16 Given $A = \begin{bmatrix} -1 & +1 \\ -100 & 99 \end{bmatrix}$

Consider two system of linear equations.

$Ax = b$, $\quad A(x + \delta x) = b + \delta b$

(i) Create an example by giving $x, \delta x, b, \delta b$

s.t $\dfrac{\|\delta x\|_\infty}{\|x\|_\infty}$ is small

but $\dfrac{\|\delta b\|_\infty}{\|b\|_\infty}$ is large

(ii) $\dfrac{\|\delta x\|_\infty}{\|x\|_\infty}$ is large and $\dfrac{\|\delta b\|_\infty}{\|b\|_\infty}$ is small.

Q.17 Given $A \in \mathbb{R}^{m\times n}$, (with $m \geq n$) Let Rank $(A) = n$

Let $B = A^T A$. Show that B is positive definite

Let $B = AA^T$ Show that B is positive semi-definite