Thm: Let A have n distinct eigenvalues with n- linearly independent eigenvectors as specified below.

eigenvalues	λ_1	λ_2	$\cdots$	λ_n.
eigenvectors Right	x_1	x_2	$\cdots$	x_n
" left	y_1^*	y_2^*	$\cdots$	y_n^*

Then $y_j^* x_i \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases}$

Proof: Consider a right eigenvector x_j

$$A x_j = \lambda_j x_j$$

Now let y_i^* be another left eigenvector.

then $y_i^* A x_j = \lambda_j y_i^* x_j$

$\Rightarrow \lambda_i \cancel{A} y_i^* x_j = \lambda_j y_i^* x_j$

$\Rightarrow$ either $\lambda_i = \lambda_j$ or $y_i^* x_j = 0$

or both.

$\lambda_i \neq \lambda_j$ by hypothesis of thm.

So $y_i^* x_j = 0$. is true. for $y_i^* \neq 0$, $x_j \neq 0$.

Now. let $y_i^* x_j = 0$ for $i = j$.

then. $y_i^* x_1 = 0$,

$y_i^* x_2 = 0$

$\vdots$

$y_i^* x_n = 0$

$\Rightarrow y_i^* \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = 0$

Since $x_1 \cdots x_n$ are linear ind. $y_i = 0$

Contradiction to $y_i^* \neq 0$ Thus proved.

Condition ~~number~~ of simple eigenvalue

Sensitivity of an eigenvalue.

Let λ be a simple eigenvalue of $A \in \mathbb{C}^{n\times n}$.

~~x~~ also let

$$Ax = \lambda x \qquad x \text{ is right eigenvector}$$

$$y^* A ~~y~~ = \lambda y^* \qquad y^* \text{ is left eigenvector.}$$

with $\|x\|_2 = \|y\|_2 = 1$

~~If~~ let $Y^{*} A X = J$ is Jordan decomposition.

~~then~~ $Y^* = X^{-1}$

$$Y^* X = I$$

in X and Y^* there is a column i with

$$X(:,i) = x, \quad Y^*(i,:) = y^*.$$

then $y^* x = 1$

Now. let A be perturbed in the direction of F by ϵ: $(A + \epsilon F)$. Also let $\|F\|_2 = 1$.

let $\lambda(\epsilon)$ be changed eigenvalue.

$$(A + \epsilon F)x(\epsilon) = \lambda(\epsilon)\, x(\epsilon) \qquad \|F\|_2 = 1$$

$$\frac{d}{d\epsilon}\left[(A + \epsilon F)\, x(\epsilon)\right] = \frac{d\lambda}{d\epsilon} x(\epsilon) + \lambda(\epsilon) \frac{d x(\epsilon)}{d\epsilon}.$$

$$F x(\epsilon) + (A + \epsilon F)\frac{d}{d\epsilon} x(\epsilon) = x(\epsilon)\frac{d\lambda}{d\epsilon} + \lambda(\epsilon)\frac{dx}{d\epsilon}$$

~~Let~~ Now $\lambda(0) = \lambda, \quad x(0) = x.$

then.

$$A\dot{x}(0) + Fx = \dot{\lambda}(0)x + \lambda\dot{x}(0)$$

$$\cancel{y^* A\dot{x}(0)} + y^* F x = \dot{\lambda}(0) y^* x + \cancel{\lambda y^* \dot{x}(0)}$$

$$|\dot{\lambda}(0)| = \left|\frac{y^* F x}{y^* x}\right| \leq \frac{1}{|y^* x|}$$

since $\|F\|_2 = 1$.

$$|y^* F x| \leq \|y^*\|_2 \|Fx\|_2$$

$$\leq \|F\|_2 \|x\|_2 = 1$$

(equality achieved if $F = yx^*$)

$s(\lambda) := |y^* x| = |\cos(\theta_{xy})|$

for a small $\epsilon > 0$ we thus get.

$$|\lambda(\epsilon) - \lambda(0)| \leq \frac{\epsilon}{|y^* x|}$$

so if $|y^* x|$ is small the

eigenvalue λ is ill conditioned.

in the direction $F = yx^*$.

Thm: If λ is distinct and $s(\lambda) < 1$, then $\exists\, \delta A \in \mathbb{R}^{n\times n}$ s.t λ is repeated eigenvalue of $A + \delta A$ and

$$\frac{\|\delta A\|_2}{\|A\|_2} \leq \frac{s(\lambda)}{\sqrt{1 - s(\lambda)^2}} = \cot(\theta_{xy})$$

Sensitivity of eigenvectors

Let T be a block upper triangular matrix

$$T = \left[\begin{array}{c|c} \lambda & w^T \\ \hline 0 & \hat{T} \end{array}\right]$$

$\lambda(\hat{T}) \cap \lambda = \phi$

λ is simple eigenvalue of T.

$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is eigenvector corresponding to λ.

Special Perturbation. T is perturbed only at zero part.

$$T + \delta T = \begin{bmatrix} \lambda & w^T \\ y & \hat{T} \end{bmatrix}, \quad \text{st } \|y\|_2 \text{ is small}$$

and let $\dfrac{\|\delta T\|_2}{\|T\|_2} = \dfrac{\|y\|_2}{\|T\|_2} = \epsilon \ll 1$

for small ϵ, $T + \delta T$, has $\lambda + \delta\lambda$ as an eigenvalue associated to eigenvector $\begin{bmatrix} 1 \\ z \end{bmatrix}$ near e_1

We want to find k s.t. $\|z\|_2 \le k\epsilon$, this k will be

the condition number of eigenvector.

$$\begin{bmatrix} \lambda & w^T \\ y & \hat{T} \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ z \end{bmatrix} (\lambda + \delta\lambda)$$

$$\lambda + w^T z = \lambda + \delta\lambda \quad , \Rightarrow \quad w^T z = \delta\lambda.$$

$$y + \hat{T} z = \lambda z + \delta\lambda z$$

$$(\hat{T} - \lambda I) z = -y + \delta\lambda z = -y + \underbrace{(w^T z) z}_{O(\epsilon^2)}$$

$$z = -(\hat{T} - \lambda I)^{-1} y + (\hat{T} - \lambda I)^{-1} (w^T z) z.$$

$$\|z\|_2 \le \|(\hat{T} - \lambda I)^{-1}\| \, \|y\|_2 + O(\epsilon^2)$$

$$\le \underbrace{\|(\hat{T} - \lambda I)^{-1}\|_2 \, \|T\|_2}_{k} \epsilon + O(\epsilon^2)$$

$k := \|(\hat{T} - \lambda I)^{-1}\|_2 \, \|T\|_2$ is the condition number of e_1 w.r.t. δT of the special form.