Practice Set 2.

Q.1.

$\mathcal{B}_s = \{1, x, x^2, x^3\}$ is the set of basis elements for polynomials of degree ≤ 3.

$p(x) \in P_3$ is $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$= \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$\cancel{\frac{d}{dx} p(x)}$ $\quad \frac{d}{dx} : P_3 \longrightarrow P_3$

$p(x) \longmapsto p'(x)$

to express $\frac{d}{dx}$ in matrix form.

Consider action of $\frac{d}{dx}$ on each element of the basis and express the result again interms of basis elements.

$\frac{d}{dx} 1 = 0 = 0(1) + 0(x) + 0(x^2) + 0(x^3)$

$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\frac{d}{dx} x = 1 = 0(1) + 0x + 0(x^2) + 0(x^3) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\frac{d}{dx} x^2 = 2x = 0(1) + 2(x) + 0(x^2) + 0(x^3) = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

$\frac{d}{dx} x^3 = 3x^2 = 0(1) + 0(x) + 2(x^2) + 0(x^3) = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$

$\text{mat}\left(\frac{d}{dx}\right) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, Check $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix}$

Similarly for $\{1, x, 2x^2-1, 4x^3-3x\}$

$$\frac{d}{dx}1 = 0(1) + 0(x) + 0(2x^2-1) + 0(4x^3-3x)$$

$$\frac{d}{dx}x = 1 = 1(1) + 0(x) + 0(2x^2-1) + 0(4x^3-3x)$$

$$\frac{d}{dx}(2x^2-1) = 4x = 0(1) + 4(x) + 0(2x^2-1) + 0(4x^3-3x)$$

$$\frac{d}{dx}(4x^3-3x) = 12x^2-3 = 3(1) + 0(x) + 6(2x^2-1) + 0(4x^3-3x)$$

$$\text{mat}\left(\frac{d}{dx}\right) \text{ in this basis is } \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

From this example, one sees that a linear map matrix representation is basis dependent.

Q. 2

Choose basis to be

$$\{b,\ Ab,\ A^2b,\ A^3b,\ A^4b\}.$$

Also.

$$A^5 = -A^4 - A^3 - A^2 - A - I$$

Thus,

$$A^5b = -A^4b - A^3b - A^2b - Ab - b.$$

In basis chosen above.

A acting on b.

$$Ab = 0b + 1Ab + 0A^2b + 0A^3b + 0A^4b$$

$$A(Ab) = 0b + 0Ab + 1A^2b + 0A^3b + 0A^4b$$

$$A(A^2b) = 0b + 0Ab + 0A^2b + 1A^3b + 0A^4b$$

$$A(A^3b) = 0b + 0Ab + 0A^2b + 0A^3b + 1A^4b$$

$$A(A^4b) = -b - Ab - A^2b - A^3b - A^4b$$

Thus mat(A) in this basis is

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Q.3 Since the matrix is in Block upper triangular form. a space $\mathcal{W} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

forms an A-invariant space.

As a side question try to find A-invariant Subspace for following matrix.

$$A = \left[\begin{array}{ccc|ccc|ccc} 1/\sqrt{2} & 501 & 195 & 0 & 0 & 0 & 25 & 100 & 4 \\ 1001 & 205 & 10^{10} & 0 & 0 & 0 & 2 & 5 & 6 \\ 0 & 111 & 10^{100} & 0 & 0 & 0 & 1 & 2 & 11 \\ \hline 0 & 0 & 0 & 1/\sqrt{2} & \pi & e & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\pi} & \pi e & e^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^2 & e & 1 & 0 & 0 & 0 \\ \hline 25 & 100 & 4 & 0 & 0 & 0 & 6 & 60 & 6000 \\ 2 & 6 & 5 & 0 & 0 & 0 & 10 & 2 & 101 \\ 2 & 1 & 11 & 0 & 0 & 0 & 1011 & 11 & 125 \end{array}\right]$$

Q.4 Note that if $Ax = \lambda x$ for some vector $x \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$

then $A^2 x = \lambda A x = \lambda^2 x$

Similarly $A^k x = \lambda^k x$.

Also note that if x is an eigenvector of A and B corresponding to eigenvalues λ_1 and λ_2 respectively then 1

$\lambda_1 + \lambda_2$ is eigenvalue of $A + B$ corresponding eigenvector is x.

Reachable space.

Q. 1

$$W = \text{Span}\left\{ \begin{bmatrix} 0 \\ -1/m_1 \\ 0 \\ -1/m_2 \end{bmatrix}, \begin{bmatrix} -1/m_1 \\ 0 \\ -1/m_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{+K}{m_1^2} - \frac{K}{m_1 m_2} \\ 0 \\ \frac{K}{m_2^2} - \frac{K}{m_1 m_2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{+K^2}{m_1^3} + \frac{K}{m_1^2 m_2} - \cdots \\ 0 \\ \cdots \end{bmatrix} \right\}$$

Note that $W = \mathbb{R}^4$.

However if $m_1 = m_2 = 1$, $K = 1$

then $W = \text{Span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^4$

Extend the basis to $\mathbb{R}^4$

$$\begin{array}{cccccc} & & e_1 & e_2 & e_3 & e_4 \\ \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}, & \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \end{array}$$

Use Steinetz procedure for extension of basis. We had discussed it in the class once.

$$\left[\begin{array}{cc|cccc} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

using pivots from only first two columns makes zero columns

$$\left[\begin{array}{cc|cccc} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cccc} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

Since none of the columns became zero any two vectors from e_1, e_2, e_3, e_4 can be added

Let us add e_1 , e_2

$$T = \begin{array}{c} \begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \end{array} \\ \left[\begin{array}{cc|cc} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

find A_{11}, A_{12}, A_{22}, B_1 s.t

$$T^{-1}AT = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]$$

$$T^{-1}B = \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right]$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

$$Aq_1 = A \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (0q_1 + 1q_2 + 0q_3 + 0q_4)$$

$$Aq_2 = A \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0[\ ] + 0[\ ] + 0[\ ] + 0[\ ] = 0q_1 + 0q_2 + 0q_3 + 0q_4$$

$$Aq_3 = A \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 1q_1 + 0q_2 + 0q_3 + 0q_4$$

$$Aq_4 = A \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} +1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0q_1 + 0q_2 + 1q_3 + 0q_4$$

$$T^{-1}AT = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \Big\} B_1$$