

1 Tutorial Problems

Q 1.1 Find out the differential equation for the circuit shown in Figure 1.

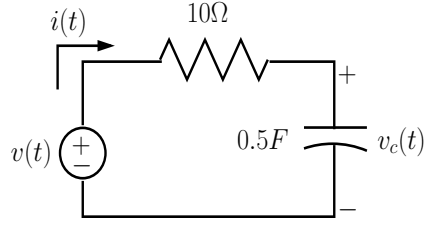


Figure 1: Figure for Q 1.1

Solve the differential equation considering initial voltage across the capacitor, $v_c(0-)$, and input voltage to be $5u(t)$, step function of 5 Volts. Plot $v(t)$ and $v_c(t)$ on the same graph.

Solution 1.1 Applying KVL round the loop we get that

$$10i + v_c = v.$$

Now, using the fact that $i = 0.5 \frac{dv_c}{dt}$ we get

$$10 \times 0.5 \frac{dv_c}{dt} + v_c = v \Rightarrow 5 \frac{dv_c}{dt} + v_c = v.$$

The differential equation we have to solve is $5 \frac{dv_c}{dt} + v_c = 5u(t)$, which after a little modification becomes

$$\frac{dv_c}{dt} + 0.2v_c = u(t).$$

We first solve the corresponding homogeneous equation

$$\frac{dv_c}{dt} + 0.2v_c = 0.$$

The auxiliary equation of this homogeneous differential equation is

$$D + 0.2 = 0 \Rightarrow D = -0.2.$$

Therefore, the solution of the homogeneous equation is

$$v_{c, Hom}(t) = ke^{-0.2t},$$

where k is a real constant. It easily follows that $v_c(0) = k$. Thus

$$v_{c, Hom}(t) = v_c(0)e^{-0.2t}.$$

In order to get the complete solution of the differential equation we need to find out a particular solution with presence of the forcing function $u(t)$. A particular solution can be found out by the following integral

$$\begin{aligned} v_{c, Part}(t) &= \int_0^t e^{-0.2(t-\tau)} u(\tau) d\tau \\ &= e^{-0.2t} \int_0^t e^{0.2\tau} d\tau \\ &= \frac{e^{-0.2t}}{0.2} [e^{0.2\tau}]_{\tau=0}^{\tau=t} \\ &= 5(1 - e^{-0.2t}). \end{aligned}$$

Thus the complete solution of the differential equation is given by

$$v_c(t) = v_{c, Hom}(t) + v_{c, Part}(t) = v_c(0)e^{-0.2t} + 5(1 - e^{-0.2t}).$$

Figure 2 shows $v(t)$ and $v_c(t)$ plotted on same scale.

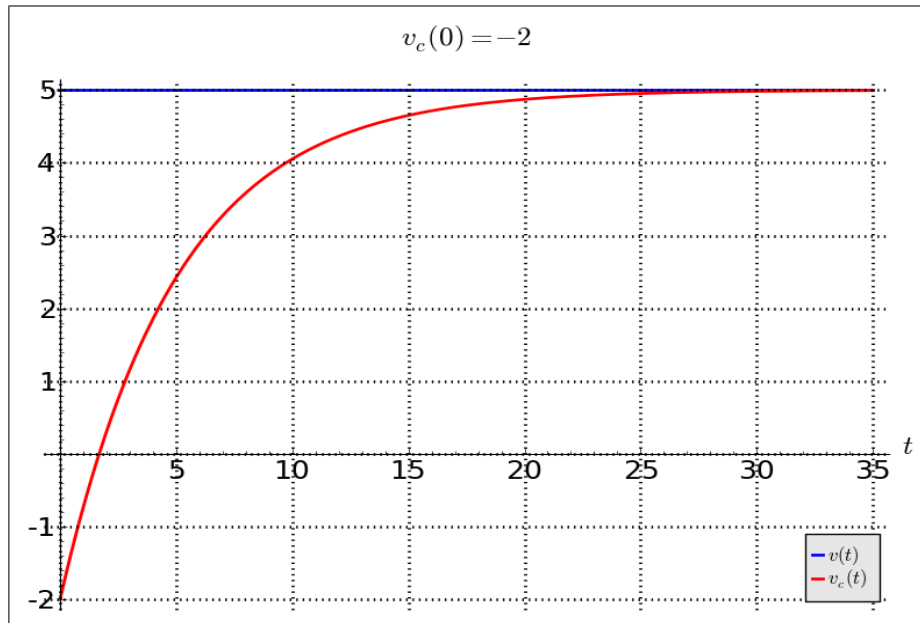


Figure 2: $v(t)$ and $v_c(t)$ plotted on same scale

Q 1.2 Solve the following differential equations. Indicate the natural response part and the forced response part in each of the solutions.

- (a) $\frac{dx}{dt} + 7x = 2(\sin 10t)u(t)$, with $x(0) = 5$.
 (b) $\frac{d^2x}{dt^2} + 4x = u(t)$, with $x(0) = 1$ and $\dot{x}(0) = 0$.
 (c) $\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0$, with $x(0) = 1$ and $\dot{x}(0) = 1$.

Solution 1.2 (a) The given differential equation is $\frac{dx}{dt} + 7x = 2(\sin 10t)u(t)$, with $x(0) = 5$. We first solve the homogeneous equation to obtain the natural response part. The homogeneous equation is

$$\frac{dx}{dt} + 7x = 0.$$

The corresponding auxiliary equation is

$$D + 7 = 0 \Rightarrow D = -7.$$

Therefore, the solution of the homogeneous part is

$$x_{Hom}(t) = ke^{-7t}.$$

Putting the initial condition, $x(0) = 5$ in the equation above, we get

$$k = 5.$$

Thus,

$$x_{Hom}(t) = 5e^{-7t}.$$

The forced response is given by the following integral

$$\begin{aligned}
 x_{Part}(t) &= \int_0^t e^{-7(t-\tau)} 2 \sin(10\tau) u(\tau) d\tau \\
 &= 2e^{-7t} \int_0^t e^{7\tau} \sin(10\tau) d\tau \\
 &= 2e^{-7t} \frac{49}{149} \left[e^{7\tau} \left(\frac{1}{7} \sin 10\tau - \frac{10}{49} \cos 10\tau \right) \right]_{\tau=0}^{\tau=t} \\
 &= 2e^{-7t} \frac{49}{149} \left[e^{7t} \left(\frac{1}{7} \sin 10t - \frac{10}{49} \cos 10t \right) + \frac{10}{49} \right] \\
 &= \frac{14}{149} \sin 10t - \frac{20}{149} \cos 10t + \frac{20}{149} e^{-7t}.
 \end{aligned}$$

Thus, the complete solution is given by

$$x(t) = x_{Hom}(t) + x_{Part}(t) = \underbrace{5e^{-7t}u(t)}_{\text{natural response}} + \underbrace{\left[\frac{14}{149} \sin 10tu(t) - \frac{20}{149} \cos 10tu(t) + \frac{20}{149} e^{-7t}u(t) \right]}_{\text{forced response}}.$$

- (b) The given differential equation $\frac{d^2x}{dt^2} + 4x = u(t)$, with $x(0) = 1$ and $\dot{x}(0) = 0$. We first solve the homogeneous equation to obtain the natural response part. The homogeneous equation is

$$\frac{d^2x}{dt^2} + 4x = 0.$$

The corresponding auxiliary equation is

$$D^2 + 4 = 0 \Rightarrow D = \pm j2.$$

Therefore, the solution of the homogeneous part is

$$x_{Hom}(t) = k_1 \cos 2t + k_2 \sin 2t.$$

Putting the initial condition, $x(0) = 1$ in the equation above, we get

$$k_1 = 1.$$

And, putting the initial condition, $\dot{x}(0) = 0$ in the equation above, we get

$$2k_2 = 0 \Rightarrow k_2 = 0.$$

Thus,

$$x_{Hom}(t) = \cos 2t.$$

The forced response is given by the following integral

$$\begin{aligned}
 x_{Part}(t) &= \frac{1}{2} \int_0^t \sin(2(t-\tau)) u(\tau) d\tau \\
 &= -\frac{1}{4} [\cos 2\tau]_{\tau=0}^{\tau=t} \\
 &= \frac{1}{4} - \frac{1}{4} \cos 2t.
 \end{aligned}$$

Thus, the complete solution is given by

$$x(t) = x_{Hom}(t) + x_{Part}(t) = \underbrace{\cos 2tu(t)}_{\text{natural response}} + \underbrace{\frac{1}{4}(1 - \cos 2t)u(t)}_{\text{forced response}}.$$

- (c) The given differential equation is $\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0$, with $x(0) = 1$ and $\dot{x}(0) = 1$. We first solve the homogeneous equation to obtain the natural response part. The homogeneous equation is

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0.$$

The corresponding auxiliary equation is

$$D^2 + D + 1 = 0 \Rightarrow D = \frac{-1 \pm j\sqrt{3}}{2}.$$

Therefore, the solution of the homogeneous part is

$$x_{Hom}(t) = e^{-t/2} \left(k_1 \cos \sqrt{3}t/2 + k_2 \sin \sqrt{3}t/2 \right) u(t).$$

Putting the initial condition, $x(0) = 1$ in the equation above, we get

$$k_1 = 1.$$

And, putting the initial condition, $\dot{x}(0) = 0$ in the equation above, we get

$$-\frac{k_1}{2} + \frac{\sqrt{3}k_2}{2} = 0 \Rightarrow k_2 = 1/\sqrt{3}.$$

Thus,

$$x_{Hom}(t) = e^{-t/2} \left(\cos \sqrt{3}t/2 + \frac{1}{\sqrt{3}} \sin \sqrt{3}t/2 \right) u(t).$$

The forced response is zero in this case because the forcing function is zero.

$$x_{Part}(t) = 0.$$

So the complete solution is

$$x(t) = x_{Hom}(t) + x_{Part}(t) = e^{-t/2} \left(\cos \sqrt{3}t/2 + \frac{1}{\sqrt{3}} \sin \sqrt{3}t/2 \right) u(t).$$

Q 1.3 Solve the following differential equations assuming all initial conditions to be zero. Indicate the transient part and the steady state part in each of the solutions.

- (a) $\frac{dx}{dt} + x = (1 + t + t^2)u(t)$.
 (b) $\frac{dx}{dt} + 7x = u(t) - u(t - 1)$.
 (c) $\frac{d^2x}{dt^2} + 25x = e^{-t}u(t)$.

Solution 1.3 (a) The given differential equation is $\frac{dx}{dt} + x = (1 + t + t^2)u(t)$. Since the initial condition is zero, we need to find out only the forced response. Since the auxiliary equation $D + 1 = 0$ has the root $D = -1$, the forced response is given by the following:

$$\begin{aligned} x(t) &= \int_0^t e^{-(t-\tau)} (1 + \tau + \tau^2) u(\tau) d\tau \\ &= e^{-t} \int_0^t e^{\tau} (1 + \tau + \tau^2) d\tau \\ &= e^{-t} \left([e^{\tau}]_{\tau=0}^{\tau=t} + [\tau e^{\tau} - e^{\tau}]_{\tau=0}^{\tau=t} + [\tau^2 e^{\tau} - 2\tau e^{\tau} + 2e^{\tau}]_{\tau=0}^{\tau=t} \right) \\ &= e^{-t} (t^2 e^t - t e^t + 2e^t - 2) \\ &= \underbrace{(t^2 - t + 2)u(t)}_{\text{steady-state}} - \underbrace{2e^{-t}u(t)}_{\text{transient}}. \end{aligned}$$

- (b) The given differential equation is $\frac{dx}{dt} + 7x = u(t) - u(t-1)$. Since the initial condition is zero, we need to find out only the forced response. Since the auxiliary equation $D + 7 = 0$ has the root $D = -7$, the forced response is given by the following:

$$\begin{aligned}
 x(t) &= \int_0^t e^{-7(t-\tau)}(u(\tau) - u(\tau-1))d\tau \\
 &= e^{-7t} \int_0^t e^{7\tau} d\tau - e^{-7t} \int_1^t e^{7\tau} d\tau \\
 &= e^{-7t} \left(\left[\frac{1}{7} e^{7\tau} \right]_{\tau=0}^{\tau=t} - \left[\frac{1}{7} e^{7\tau} \right]_{\tau=1}^{\tau=t} \right) \\
 &= \frac{1}{7} (1 - e^{-7t}) u(t) - \frac{1}{7} (1 - e^{-7(t-1)}) u(t-1) \\
 &= \underbrace{\frac{1}{7}(u(t) - u(t-1))}_{\text{steady state}} - \underbrace{\frac{1}{7}(e^{-7t}u(t) - e^{-7(t-1)}u(t-1))}_{\text{transient}}.
 \end{aligned}$$

- (c) The given differential equation is $\frac{d^2x}{dt^2} + 25x = e^{-t}u(t)$. Since the initial condition is zero, we need to find out only the forced response. The forced response is given by the following:

$$\begin{aligned}
 x(t) &= \int_0^t 5 \sin(5(t-\tau))e^{-\tau}u(\tau)d\tau \\
 &= \underbrace{-\frac{25}{26} \cos(5t) + \frac{5}{26} \sin(5t)}_{\text{steady state}} + \underbrace{\frac{25}{26} e^{-t}}_{\text{transient}}.
 \end{aligned}$$

Q 1.4 Assuming $x(0) = -1$ and $\dot{x}(0) = 1$, solve $\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = f(t)$ for following two cases namely

- (a) $f(t) = u(t)$
 (b) $f(t) = 1$ for all $t \in (-\infty, \infty)$

Solution 1.4 In solution to question ??(c), the homogeneous solution was found to be

$$x_{Hom}(t) = e^{-t/2} \left(k_1 \cos \sqrt{3}t/2 + k_2 \sin \sqrt{3}t/2 \right).$$

- (a) Substituting the initial condition, $x(0) = 1$ in the homogenous above, we get

$$k_1 = 1.$$

And, substituting the initial condition, $\dot{x}(0) = 0$ in the equation above, we get

$$-\frac{k_1}{2} + \frac{\sqrt{3}k_2}{2} = 0 \Rightarrow k_2 = \sqrt{3}.$$

Thus,

$$x_{Hom}(t) = e^{-t/2} \left(\cos \sqrt{3}t/2 + \sqrt{3} \sin \sqrt{3}t/2 \right).$$

Particular solution is given by

$$\begin{aligned}
 x_{Part}(t) &= \int_0^t e^{-\frac{1}{2}t + \frac{1}{2}\tau} \sin \left(\frac{1}{2} \sqrt{3}(t-\tau) \right) u(\tau) d\tau \\
 &= 1 - e^{-\frac{1}{2}t} \left(\cos \left(\frac{\sqrt{3}t}{2} \right) + \frac{1}{\sqrt{3}} \sin \left(\frac{\sqrt{3}t}{2} \right) \right)
 \end{aligned}$$

So, the complete solution is

$$\begin{aligned} x(t) &= x_{Hom}(t) + x_{Part}(t) = e^{-t/2} \left(\cos \sqrt{3}t/2 + \sqrt{3} \sin \sqrt{3}t/2 \right) \\ &\quad + 1 - e^{-\frac{1}{2}t} \left(\cos \left(\frac{\sqrt{3}t}{2} \right) + \frac{1}{\sqrt{3}} \sin \left(\frac{\sqrt{3}t}{2} \right) \right) \\ &= e^{-t/2} \left(\frac{2}{\sqrt{3}} \sin \sqrt{3}t/2 \right) + 1 \end{aligned}$$

- (b) Here, by using method of undetermined coefficients, it can be shown that $x_{part}(t) = 1$. However, constants in homogenous part are yet to be determined. Therefore,

$$x_{Hom}(t) = e^{-t/2} \left(k_1 \cos \sqrt{3}t/2 + k_2 \sin \sqrt{3}t/2 \right).$$

This gives us,

$$x(t) = x_{Hom}(t) + x_{Part}(t) = e^{-t/2} \left(k_1 \cos \sqrt{3}t/2 + k_2 \sin \sqrt{3}t/2 \right) + 1$$

Now, we determine the constants as follows. For $x(0) = 1$ we have, $x(0) = k_1 + 1 = 1$. Hence, $k_1 = 0$. Also

$$\dot{x}(t) = \frac{e^{-\frac{1}{2}t}}{2} \left(\sqrt{3}k_2 \cos \left(\frac{\sqrt{3}}{2}t \right) - \sqrt{3}k_1 \sin \left(\frac{\sqrt{3}}{2}t \right) \right) - \left(k_1 \cos \left(\frac{\sqrt{3}}{2}t \right) + k_2 \sin \left(\frac{\sqrt{3}}{2}t \right) \right)$$

Thus, $\dot{x}(0) = 1$ gives us $\sqrt{3}k_2 - k_1 = 2$ and hence $k_2 = \frac{2}{\sqrt{3}}$. So complete solution is

$$x(t) = x_{Hom}(t) + x_{Part}(t) = \frac{2}{\sqrt{3}}e^{-t/2} \sin \sqrt{3}t/2 + 1$$

Q 1.5 Draw a functional block diagram of a servo position control system. The servo system has a DC motor for an actuator that produces torque and takes in DC voltage as input. The torque then moves a load the position of which is to be controlled. A load torque acts on this load as a disturbance. The loop is closed by sensing the output angle of the load and feeding it back to the actuator. Modify the system to incorporate computer based control.

Solution 1.5 Left as an exercise.

Q 1.6 Draw a functional block diagram of a flyball governor system shown in Figure 3¹. The principle of its working is as follows. As output speed of prime mover increases (decreases) the flyball move outwards (inwards) due to change in centrifugal force. This causes the beam linkage to change the throttle valve position to decrease (increase) the supply of fuel to the prime mover and thereby regulating the speed as necessary.

Solution 1.6 The functional block diagram is shown in Figure 4

Q 1.7 Consider the following block diagram (Figure 5) of a feedback system. Obtain the gain of the overall system. Show that the presence of the feedback makes the overall system less sensitive to parameter variation in the plant.

Solution 1.7 Left as an exercise.

¹By MdeVicente (Own work) [CC0], via Wikimedia Commons, http://commons.wikimedia.org/wiki/File%3ACentrifugal_governor.svg

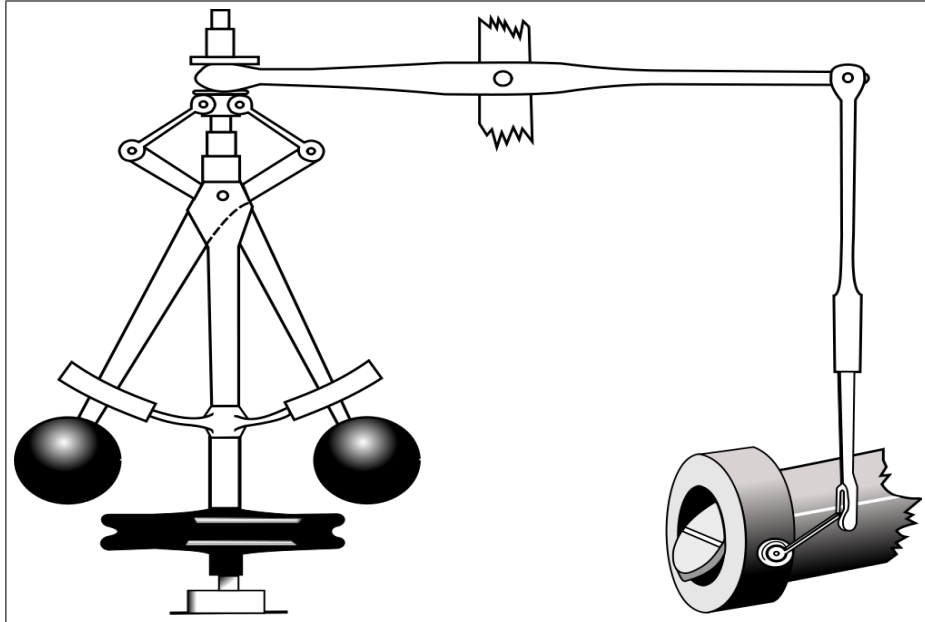


Figure 3: Flyball speed governor system

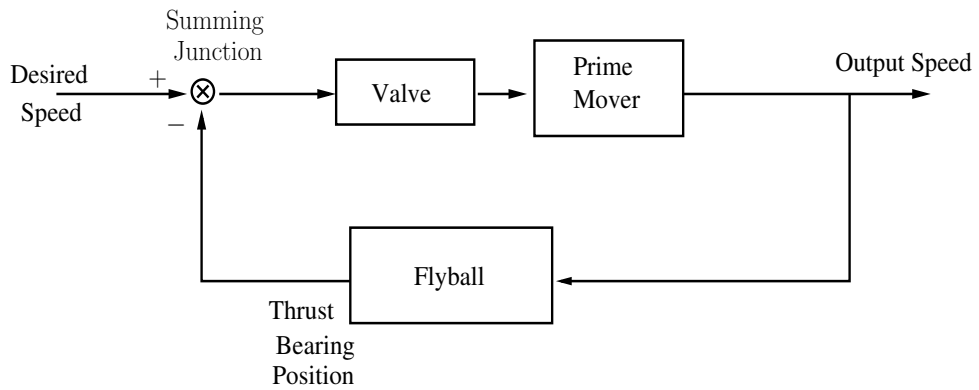


Figure 4: Functional block diagram for flyball speed governor system

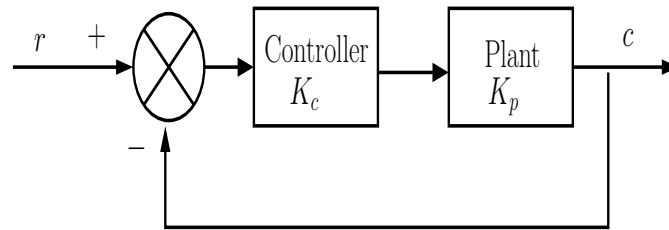


Figure 5: Figure for Q 1.7

Q 1.8 Consider a system mathematically modelled as the following differential equation.

$$\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = f(t).$$

Suppose a feedback is introduced in the system by way of imposing

$$f(t) = r(t) - kx(t),$$

where $r(t)$ is to be considered the forcing function. Write down the modified differential equation. Find out the value of the parameter K at which the system starts showing sustained oscillation for a unit step input, $u(t) = f(t)$.

Solution 1.8 The modified differential equation is

$$\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = r - kx \Rightarrow \frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + (1+k)x = r.$$

Whether the solution to unit step input has sustained oscillation or not can be checked by looking into the auxiliary equation and its roots (the characteristic roots). The auxiliary equation is

$$D^3 + 2D^2 + 2D + (1+k) = 0.$$

Sustained oscillation is possible if and only if the polynomial $D^3 + 2D^2 + 2D + (1+k)$ has purely imaginary roots. Let us put $D = j\omega$ in the polynomial. We get

$$-j\omega^3 - 2\omega^2 + 2j\omega + (1+k) = (1+k) - 2\omega^2 + j(2\omega - \omega^3).$$

In order for $j\omega$ to be a root of the polynomial we must have

$$(1+k) - 2\omega^2 = 0 \tag{1}$$

$$2\omega - \omega^3 = 0 \tag{2}$$

From equation (2) it follows $\omega = 0, \pm\sqrt{2}$. Putting $\omega = 0$ we get k must be equal to -1 , on the other hand, putting $\omega = \pm\sqrt{2}$ we get k must be equal to 3 .

Q 1.9 Find out the Laplace transforms of the following standard time functions from its definition.

- (a) Unit impulse $\delta(t)$.
- (b) Unit step $u(t)$.
- (c) Unit ramp $tu(t)$.
- (d) Exponential function $e^{-at}u(t)$ for some real number a .
- (e) Sine function $(a \sin \omega t) u(t)$.
- (f) Cosine function $(a \cos \omega t) u(t)$.

Solution 1.9 Left as an exercise.

Q 1.10 Verify the following identities using the definition of Laplace transform given by equation (2.1):

- (a) $\mathcal{L}[kf(t)] = kF(s)$ for all real number k .
- (b) $\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$.
- (c) $\mathcal{L}[e^{at}f(t)] = F(s-a)$.
- (d) $\mathcal{L}[f(t-T)u(t-T)] = e^{-sT}F(s)$ for $T > 0$.
- (e) $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$.
- (f) $\mathcal{L}\left[\int_{0-}^t f(\tau)d\tau\right] = \frac{F(s)}{s}$.

Solution 1.10 Left as an exercise.

Q 1.11 You have obtained the differential equation for the following circuit (Figure 6) in Chapter 1. Suppose the input $v(t) = 5u(t)$ in Volts, and the initial condition $v_c(0-) = -2$ Volts. Write down the differential equation after taking Laplace transform. Solve the differential equation using Laplace transform with the help of Tables 2.1 and 2.2. Plot $v(t)$ and $v_c(t)$ on the same graph.

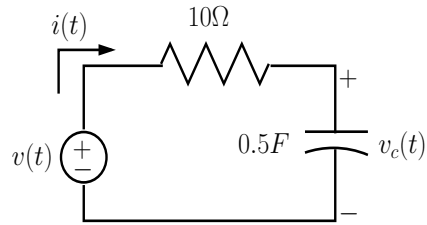


Figure 6: Figure for Q 1.11

Solution 1.11 Left as an exercise.

Q 1.12 Suppose a system is described by the following differential equation:

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = \frac{dx}{dt} + 10x.$$

Obtain the transfer function $Y(s)/X(s)$. Suppose $x(t) = \delta(t)$, the unit impulse, and solve for $y(t)$ considering all initial conditions to be zero. This solution is known as the *impulse response* of the system, and is usually denoted by $h(t)$. Suppose the input $x(t) = u(t)$, the unit step, and solve for $y(t)$ assuming all initial conditions to be zero. Show that this solution equals $h(t)$ convolved with $u(t)$.

Solution 1.12 Left as an exercise.

Q 1.13 Suppose a system is described by the following differential equation:

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 2\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 10x.$$

Obtain the transfer function $Y(s)/X(s)$. Find out the impulse response of the system assuming all initial conditions to be zero.

Solution 1.13 Left as an exercise.