Computation of SVD.

Recall for any matrix

$- \; A = U \Sigma V^T$

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r, \sigma_{r+1} \dots \sigma_n)$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_r > \sigma_{r+1} = \sigma_{r+2} \dots = \sigma_n = 0$$

How to compute SVD.

Simple way is through eigenvalue computation of symmetric matrices

$A^T A, \quad A A^T$

$$A^T A = V \Sigma U^T U \Sigma V^T$$
$$= V \Sigma^2 V^T$$

$\sqrt{\lambda(A^T A)} \to$ singular values.

$$A A^T = U \Sigma^2 U^T$$

Eigenvectors of $A^T A$ give V.

and of AA^T gives U.

But what if $\sigma_r = 0.001$

in squaring i.e. $A^T A$ $\quad \lambda_r = (0.001)^2 = 10^{-6} \approx 0$

Loss of information on squaring

Direct computation of SVD.

$A, B \in \mathbb{R}^{n \times m}$ are orthogonally equivalent

if $\exists$ $P \in \mathbb{R}^{n\times n}$ & $Q \in \mathbb{R}^{m \times m}$ s t

$$PAQ = B$$

By orthogonal equivalence transformation it is possible to reduce any matrix to bidiagonal form.

B in bidiagonal form if $b_{ij} = 0$ $i > j$ or $i < j-1$

$$\begin{bmatrix} * & * & 0 & \cdots & 0 & 0 \\ 0 & * & * & \cdots & 0 & 0 \\ 0 & 0 & * & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * & * \\ 0 & 0 & 0 & \cdots & 0 & * \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$i > j$ or.
$i < j-1$

$i - j > 0$

Only diagonal & superdiagonal are non-zero.

Let $A \in \mathbb{R}^{n \times m}$ with $n \geq m$.

$\exists\ \hat{U} \in \mathbb{R}^{n \times n}$, $\hat{V} \in \mathbb{R}^{m \times m}$ orthogonal

both are products of finite no. of

reflectors ~~and~~ s.t.

$A = \hat{U} B \hat{V}^T$ where B is in

bidiagonal form.

Golub - Kahan algorithm

Step 1: Create zeros in first column & rows of A.

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} \xrightarrow{\hat{U}_1} \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\hat{U}_1 A$ first Column has zeros except at (1, 1) place.

Let New 1st row of $\hat{U}_1 A$ $[\hat{a}_{11} \quad \hat{a}_{12} \cdots \hat{a}_{1m}]$
(i.e 1st row of $\hat{U}_1 A$)

Now we can only operate on $[\hat{a}_{12} \cdots \hat{a}_{1m}]$ without disturbing zeros in first column.

$$\hat{V}_1 = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \check{V}_1 & \\ 0 & & & \end{array}\right]$$

Where $\check{V}_1$ is s.t.

$$\left[\hat{a}_{12} \cdots \hat{a}_{1m}\right] \check{V}_1 = \left[* \; 0 \cdots 0\right]$$

Then.

$$\hat{U}_1 A \hat{V}_1 = \left[\begin{array}{c|cccc} * & * & 0 & \cdots & 0 \\ \hline 0 & & & & \\ \vdots & & \hat{A} & & \\ 0 & & & & \end{array}\right]$$

Second step.

$$\hat{U}_2 \hat{U}_1 A \hat{V}_1 \hat{V}_2 = \left[\begin{array}{c|c|cccc} * & * & 0 & 0 & \cdots & 0 \\ \hline 0 & * & * & 0 & \cdots & 0 \\ \hline 0 & 0 & & & & \\ \vdots & \vdots & & \tilde{A} & & \\ 0 & 0 & & & & \end{array}\right]$$

and so on.

$$\overbrace{\hat{U}_m \cdots \hat{U}_2 \hat{U}_1}^{\hat{U}} A \overbrace{\hat{V}_1 \hat{V}_2 \cdots \hat{V}_{m-2}}^{\hat{V}}$$

$$= \begin{bmatrix} * & * & & & \\ & * & * & & \\ & & \ddots & \ddots & \\ & & & & * \\ & & & & * \\ & & & & \end{bmatrix} = B$$

$$\hat{U} A \hat{V} = B$$

Now we can start directly with bidiagonal form.

Let $B = \begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix}$ $\tilde{B} \to \mathbb{R}^{m \times m}$. $0 \to$ doesn't matter now.

Let

$$B = \begin{bmatrix} \beta_1 & \gamma_1 & & & \\ & \beta_2 & \gamma_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \gamma_{m-1} \\ & & & & \beta_m \end{bmatrix}$$

B is properly bidiagonal if $\beta_i \neq 0$ & $\gamma_i \neq 0$

if not then $B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$ where B_1 & B_2 are properly bidiagonal

SVD problem is eig. val. problem for BB^T and B^TB

BB^T and B^TB are tridiagonal.

Let $C = \begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix}$ $\quad C^2 = \begin{bmatrix} B^TB & 0 \\ 0 & BB^T \end{bmatrix}$

C is non-singular.

all eig. values of C are ~~non-sing~~ non zero. and occur in $\pm\sigma$ pairs.

$\begin{bmatrix} \cancel{B u} \\ \cancel{B^T v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

~~u, v~~ $\begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector of C with σ as eigenvalue. then $\begin{bmatrix} u \\ -v \end{bmatrix}$ is e.vect. with $-\sigma$ as eigenvalue.

if σ is singular value of B with u, v as singular vector

then $Bu = \sigma v$
$B^Tv = \sigma u$

$\cancel{Bu = \sigma v}$
$\cancel{B^T u = \sigma v}$

$-B^Tv = -\sigma u$
$Bu = -\sigma v$

and then $\begin{bmatrix} u \\ v \end{bmatrix}$ is eigenvector of C with σ as eigenvalue, also $\begin{bmatrix} u \\ -v \end{bmatrix}$ is an eigenvector with $-\sigma$ as eigenvalue.

$$C\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} B^T v \\ B u \end{bmatrix} = \sigma \begin{bmatrix} u \\ v \end{bmatrix}$$

$$C\begin{bmatrix} u \\ -v \end{bmatrix} = \begin{bmatrix} -B^T v \\ B u \end{bmatrix} = -\sigma \begin{bmatrix} u \\ -v \end{bmatrix}.$$

Thus eigenvalue of C always occur in pair $\pm\sigma$

FRANCIS like Algorithm to obtain SVD (GOLUB REINSCH ALGORITHM)

$$C = \begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix}$$

$$\tilde{C} = P^T C P = \begin{bmatrix} 0 & \beta_1 & 0 & 0 & & & & \\ \beta_1 & 0 & \gamma_1 & 0 & & & & \\ 0 & \gamma_1 & 0 & \beta_2 & 0 & 0 & & \\ 0 & 0 & \beta_2 & 0 & \gamma_2 & 0 & & \\ & & 0 & \gamma_2 & 0 & \beta_3 & 0 & 0 \\ & & 0 & 0 & \beta_3 & 0 & \gamma_3 & 0 \\ & & & & 0 & \gamma_3 & \ddots & \\ & & & & 0 & 0 & & \ddots \end{bmatrix}$$

Eigenvalues in pairs $\pm\sigma$.

We work with $(\tilde{C} - \rho I)(\tilde{C} + \rho I)$

$$= \tilde{C}^2 - \rho^2 I$$

$$\tilde{C} - \rho I = \begin{bmatrix} -\rho & \beta_1 & 0 & 0 & & & & \\ \beta_1 & -\rho & \gamma_1 & 0 & & & & \\ 0 & \gamma_1 & -\rho & \beta_2 & 0 & 0 & & \\ 0 & 0 & \beta_2 & -\rho & \gamma_2 & 0 & & \\ & & 0 & \gamma_2 & -\rho & \beta_3 & 0 & 0 \\ & & 0 & 0 & \beta_3 & -\rho & \gamma_3 & 0 \\ & & & & 0 & \gamma_3 & & \\ & & & & 0 & 0 & & \ddots \end{bmatrix}$$

$$\tilde{C} + \rho I = \begin{bmatrix} \rho & \beta_1 & 0 & 0 & & \\ \beta_1 & \rho & \gamma_1 & 0 & & \\ 0 & \gamma_1 & \rho & \beta_2 & 0 & 0 \\ 0 & 0 & \beta_2 & \rho & \gamma_2 & 0 \\ & & 0 & \gamma_2 & \rho & \beta_3 \\ & & 0 & 0 & \beta_3 & \rho \\ & & & & & \ddots \end{bmatrix}$$

$(\tilde{C} - \rho I)(\tilde{C} + \rho I)$ first col.

$$= \begin{bmatrix} \beta_1^2 - \rho^2 \\ 0 \\ \gamma_1\beta_1 - \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{——— (6)}$$

$$V_0^T \begin{bmatrix} \beta_1^2 - \rho^2 \\ \gamma_1\beta_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

V_0

$B \to B V_0$,

$(1, 3)$ ~~column~~ n. ~~and~~ rows.

Create a zero in place of $\gamma_1\beta_1$ using rotator.

$$\begin{bmatrix} A & * & & + & & \\ * & A & * & & & \\ & * & A & * & & \\ + & & * & A & * & \\ & & & * & A & + \\ & & & & * & A \end{bmatrix}$$

⑦

① find a rotator $\tilde{Q}_0^T$ that creates zero in place of $\gamma_1 \beta_1$ in ⊛. Make it act from right on $\tilde{C}$.

$$\tilde{Q}_0^T \tilde{C} = \begin{bmatrix} 0 & \beta_1 & 0 & \sigma & \\ \beta_1 & 0 & \gamma_1 & 0 & \\ 0 & \gamma_1 & \circledast & \beta_2 & \\ 0 & \circledast & \beta_2 & 0 & \gamma_2 \\ & & & \gamma_2 & \end{bmatrix}$$

Changes only 1, 3 rows.

Combines 1, 3 rows, thus zero structure in 4^{th} column is destroyed as follows.

$$\begin{bmatrix} 0 & \tilde{\beta}_1 & 0 & \boxed{\star} & \\ \beta_1 & 0 & \gamma_1 & 0 & \\ 0 & \tilde{\gamma}_1 & 0 & \tilde{\beta}_2 & 0 \\ 0 & 0 & \beta_2 & 0 & \gamma_2 \\ \vdots & 0 & \circledast & \gamma_2 & \\ 0 & 0 & \vdots & \vdots & \end{bmatrix}$$

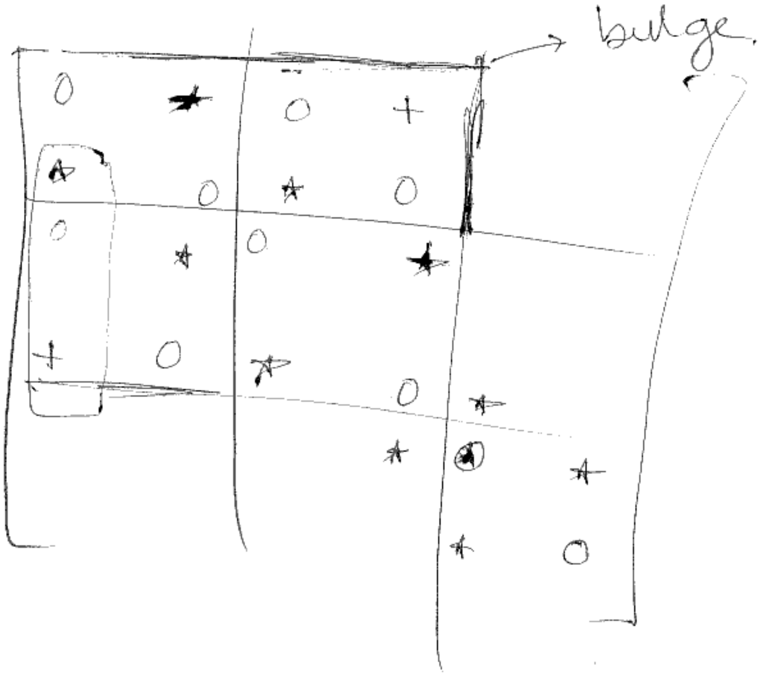
→ non-zero entry created in (4, 1) colm.

Completing Similarity transformation.

$$\begin{bmatrix} 0 & * & 0 & + & & \\ * & 0 & * & 0 & & \\ 0 & * & 0 & * & & \\ + & 0 & * & 0 & * & \\ 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & * & 0 \end{bmatrix}$$

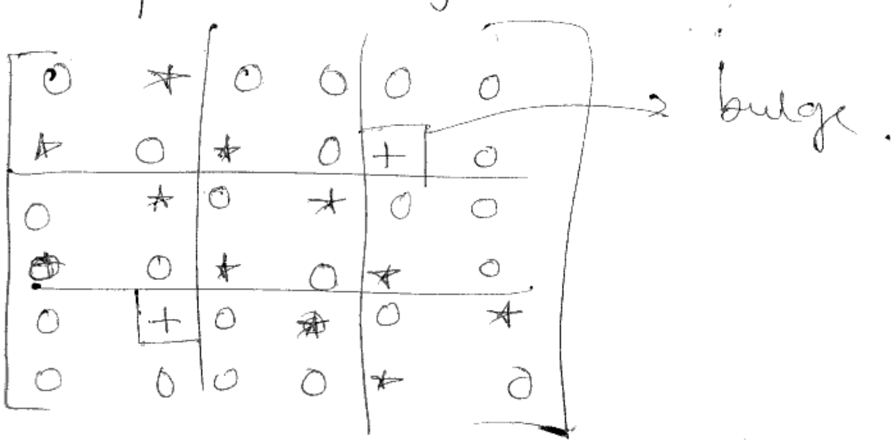
Bulge of special form is created.

$(3,1), (4,1), (4,2)$

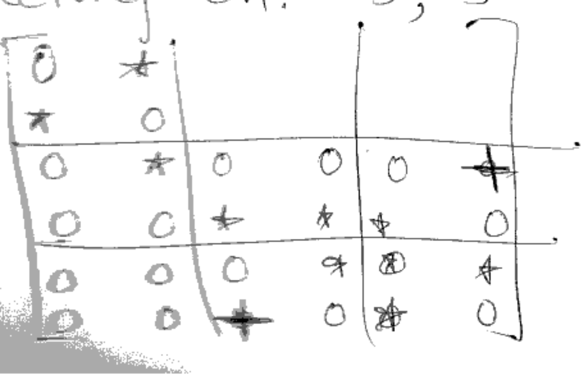


Q_1 that acts on $(2,4)$ rows.

bulge shifts as follows



Q_2 acting on. $3, 5^{th}$ rows.



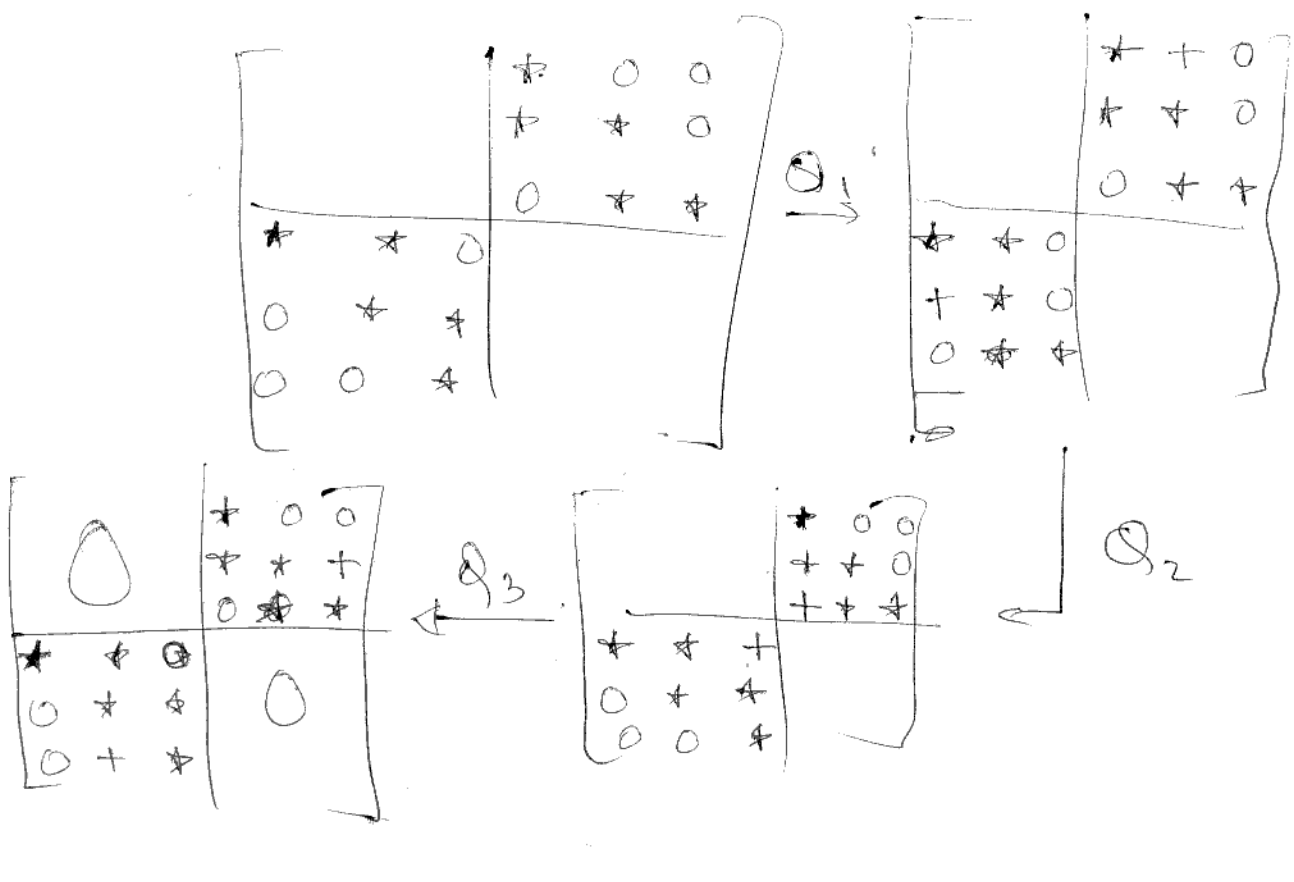
and so on.

$$\begin{bmatrix}
\ddots & & & & & \\
 & & \gamma_{m-2} & & & \\
 & \gamma_{m-2} & 0 & \beta_{m-1} & & \\
 & & \beta_{m-1} & 0 & \gamma_{m-1} & \\
 & & & \gamma_{m-1} & 0 & \beta_m \\
 & & & & \beta_m & 0
\end{bmatrix}$$

$\gamma_{m-1} \to 0$ then $\pm \beta_m$ eigenvalues.

Recall $\tilde{C} = P^T C P$.

What goes on in original unshuffled mode



$$\begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix} \underbrace{\begin{bmatrix} V & V \\ U & -U \end{bmatrix}} = \underbrace{\begin{bmatrix} V & V \\ U & -U \end{bmatrix}} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}$$

all collected orthogonal matrices in deshuffled form.

Transpose of this was ~~pre~~ being premultiplied to

$$\begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix}$$

$$\begin{bmatrix} B^T U & -B^T U \\ BV & BV \end{bmatrix} = \begin{bmatrix} \cancel{B}V\Sigma & -\cancel{B}V\Sigma \\ U\Sigma & -U\Sigma \end{bmatrix}$$

$$B^T U = V \Sigma$$

$$B V = U \Sigma$$

$$U^T B V = \Sigma$$

$$\boxed{\text{or } B = U \Sigma V^T} \quad \text{SVD.}$$