Computation of Time Optimal Feedback Control using Groebner Basis

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***Abstract*—The synthesis of time optimal *feedback* control of a single input continuous time linear time invariant (LTI) system is considered. If the control input $u(t)$ is constrained to obey $|u(t)| \leq 1$, then it is known that the optimal input switches between the extreme values ± 1, according to some "switching surfaces" in the state space. It is shown that for systems with non-zero, distinct and rational eigenvalues, these switching surfaces are semi-algebraic sets and a method to compute them using Groebner basis, is proposed. In the process the null-controllable region for such systems is characterized and computed. Numerical simulations illustrate the proposed computational methods.**

I. Introduction

The classical time optimal control transfers the states of a linear time invariant (LTI) system from a given initial state to the origin of the state space in the minimum possible time, using inputs from a constrained set of inputs (e.g. $|u| \leq 1$). The Pontryagin's maximum principle (PMP) [1], shows that the *open loop* solution to this problem is bang-bang and provides necessary conditions characterizing the optimal input. However, a feedback solution is highly desirable since (a) unlike the open loop solution, a feedback controller would not require recalculation of the entire input signal for each distinct initial condition, and (b) a feedback solution would be robust to uncertainties and disturbances and drive the system to origin repeatedly even if the system is forced away from the optimal trajectory due to external disturbances. Examples where time optimal feedback control is employed includes, space-craft attitude control [2], robotic manipulators [3] and pursuit evasion games [4], etc. In spite of several attempts (e.g. [5], [6]), a general method for constructing the feedback solution seems to be unavailable. In this article, we provide a partial solution to this synthesis problem, by providing an algorithmic method for constructing the time-optimal feedback function for controllable linear systems with non-zero, distinct and rational eigenvalues.

We consider a n^{th} order single input linear time invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(0) = x_0 \tag{1}$$

where the initial condition $x_0 \in X_0 \subseteq \mathbb{R}^n$, $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the state vector and the input $u(t)$ is constrained within a normalized set $U = \{u(t) : |u(t)| \leq 1\}$. It was shown in [1], [7] that the time optimal control switches between the extreme admissible values (± 1) according to the so called "switching surfaces" in the state space. Though, switching surfaces were constructed explicitly for some example second-order systems in [1], [7], the structure and properties of the switching surfaces were described much later by [5] and [6]. Subsequently, several articles considered the synthesis of these surfaces for limited cases e.g. for second-order LTI systems [1], [8], [9], [10] and n-integrator chains [11]. But in general there is no method to synthesize such surfaces for n^{th}-order systems.

In this article, for controllable systems with non-zero, distinct and rational eigenvalues, we show that the resulting switching surfaces can be described by semi-algebraic sets. Moreover, using a Groebner basis based implicitization algorithm [12], we derive implicit expressions, in terms of polynomials and rational functions, describing the switching surfaces. A feedback logic for optimal switching is synthesized using these polynomials of the state variables. By extension, we show that the set of initial conditions transferable to the origin is also a semi-algebraic set and algorithms for the computation of this set are provided. A preliminary version of these results was presented in [13].

The use of Groebner basis for implicitization and other applications are described in [12], [14], [15]. Some relatively recent applications of this technique in systems theory can be found in [14], [16], [17], [18], [19]. All the Groebner basis computations in this paper were done with [20], [21].

II. Terminal Manifolds and Switching Logic

The theory of terminal manifolds and switching locus [5] is reviewed briefly in this section. Assume that the A matrix in the LTI system described by (1) has non-zero, distinct and rational eigenvalues ($\lambda(A) \in \mathbb{Q} - \{0\}$), and the pair $\{A, B\}$ is controllable. The set of states of system (1) that can be steered to origin is characterized by $x_0 = -\int_0^t e^{-A\tau} Bu(\tau) d\tau$. If the input $u(t)$ is allowed to vary over the set of admissible inputs U, then x_0 defines all initial conditions which can be steered to the origin in time t. Such initial conditions are called null-controllable states in time t and they are characterized by the following set: $R(t) = \{x : x = \int_0^t e^{-A\tau} Bu(\tau) d\tau, \ \forall u(t) \in U\}$. The set of all null-controllable states is, $X_0 = \bigcup_{t \in [0,\infty)} R(t)$. The set X_0 can be written in a parametric form by using the following theorem [5].

Theorem 1. *For any $k = 1, ..., n$ and a sequence $0 < t_1 < ... < t_k < \infty$, consider the control u on $[0, t_k]$ with values ± 1 alternating in intervals $[0, t_1), ..., [t_{k-1}, t_k)$. Then u is an optimal control, so that the point, $x = \pm \left(-\int_0^{t_1} + \int_{t_1}^{t_2} - ... + (-1)^k \int_{t_{k-1}}^{t_k} \right) e^{-A\tau} B d\tau$ has t_k as the least time required to reach origin. Conversely, every optimal control on any interval $[0, \theta]$ where $0 < \theta < \infty$ is of the described form.*

To characterize all states which can be steered to origin in time t_k using a bang-bang input with k-switches, we define the following functions:

$$\begin{aligned} F_k^+(t_1, ..., t_k) &= \left(-\int_0^{t_1} + ... + (-1)^k \int_{t_{k-1}}^{t_k} \right) e^{-A\tau} B d\tau \\ F_k^-(t_1, ..., t_k) &= -F_k^+(t_1, ..., t_k) \end{aligned} \tag{2}$$

Observe that there is a physical binding on $t_i, \quad i = 1, 2, ..., k$ i.e $0 \leq t_1 \leq t_2 \leq ... \leq t_k$. Let, $V_k = \{(t_1, t_2, ..., t_k) :$

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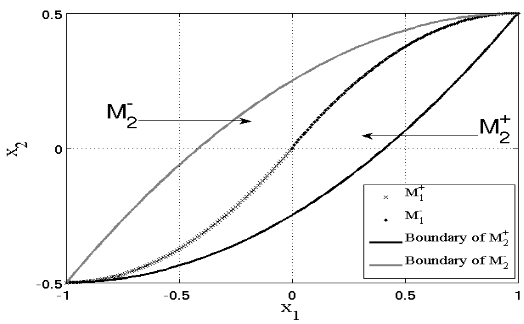

Fig. 1. $X_0 = M_2^+ \cup M_2^-$ for case of example 4

$0 \le t_1 \le t_2 \le ... \le t_k < \infty\}$ for $k = 1, ..., n$. Then the set of states, which can be steered to origin in less than k-switches with $u = +1$ to begin with, is denoted by M_k^+ and can be defined as follows: $M_k^+ = \{x : x = F_k^+(v), \forall v \in V_k\}$. Similarly we can define $M_k^- = -M_k^+$. Thus the set of all states which can be steered to origin in $n-1$ switches can be defined as follows: $M_n = \{x : x = F_n^{\pm}(v),\ \forall v \in V_n\}$. The following results hold [5]:

Corollary 2. *For a system defined by (1), a bang-bang control on an interval* $[0, \infty)$ *is time-optimal iff it has at most* $n-1$ *discontinuities.*

Lemma 3. $M_n = X_0$ *[5].*

Lemma 3 is extremely useful for our purposes, since it states that the null-controllable region with only bang-bang inputs (with at most $n-1$ switches) is actually the entire null controllable region. Hence any state in X_0 can be driven to the origin with such a bang-bang control input. Moreover, such an input is always time optimal.

It is further shown in [5], that the set X_0 is divided symmetrically into two parts M_n^+ and M_n^- by M_{n-1}, which itself is divided into two parts M_{n-1}^+ and M_{n-1}^- by M_{n-2} and so on. In general we can write, $M_k = M_k^+ \cup M_k^- \forall k = 1, ..., n$ and $X_0 = M_n^+ \cup M_n^-$. Clearly, $M_0 \subset M_1 \subset ... \subset M_n$. We illustrate the structure of M_k for a simple 2-state system:

Example 4. For a second order LTI system $\dot{x} = Ax + Bu$ with $A = \text{diag}(1, 2)$, $B = [1\ 1]^T$ and $|u| \le 1$, the corresponding structure of X_0, which is divided in two parts namely M_2^+ and M_2^- by $M_1^+ \cup M_1^-$ is as shown in figure 1.

Based on this structure of X_0 an iterative switching logic can be defined that drives any state in X_0 to the origin in minimum time. For any initial condition $x_0 \in \text{int } X_0$, time-optimal switching should ideally induce the following set of events. We assume $x_0 \in \text{int } M_n^+$ for illustration. A similar sequence would be valid for $x_0 \in \text{int } M_n^-$ with opposite signs:

- Input $u = 1$ pushes x from M_n^+ to the manifold M_{n-1}^-.
- As soon as $x \in M_{n-1}^-$, switch 2 pushes x to M_{n-2}^+ and so on.
- Input at $(n-1)^{th}$ switch pushes x from M_1^+(if n is odd) or M_1^-(if n is even) to the origin.

On inspection of this logic, it seems that we need to compute M_k^+ and $M_k^-(k = 1, ..., n-1)$ to implement feedback based switching. Before we give an algorithm for this computation in the next section, we note the following:

Remark 5. Recall that A was assumed to have distinct rational eigenvalues. Without loss of generality, we can further assume A to be diagonal, for the construction of the switching surfaces M_k $(k = 1, ..., n-1)$ and for using the switching logic defined above. This is because, for any real similarity transformation $\widehat{x} = Tx$ on a system, the corresponding set $\widehat{M}_k^+ = \{\widehat{x} = Tx :\ x \in M_k^+\}$ and similarly $\widehat{M}_k^- = \{\widehat{x} = Tx :\ x \in M_k^-\}$ [5]. Thus, computation of M_k^+ and M_k^- for the diagonalized system is enough to compute the corresponding $\widehat{M}_k^+$ and $\widehat{M}_k^-$ for all similar systems.

III. Parametric Representation

In this section, we show, using a simple substitution of variables, that the switching surfaces M_k , $k = 1, ..., n-1$ and the null-controllable set $X_0 = M_n$ can be represented parametrically as polynomials. These polynomials are then implicitized in sections IV and V, using Groebner basis techniques to get polynomial equalities and inequalities involving only the state variables. These semi-algebraic sets define both the switching surfaces and the null-controllable region.

Recall that all $x \in M_k$ are characterized by the functions F_k^+ or F_k^- defined in (2), which takes $t_1, ..., t_k$ as arguments. Since we can assume A to be diagonal without loss of generality (see section II: remark 5 above), we can alternatively write each component of state x $(x_i, i = 1, .., n)$ as some other function, denoted as $f_{ki}^{\pm}$, which takes arguments $e^{-\lambda_i t_1}, ..., e^{-\lambda_i t_k}$ for all $i = 1, ..., n$. Here f_{ki}^+ corresponds to F_k^+ and f_{ki}^- to F_k^-. Thus, $x_i = f_{ki}^{\pm}(e^{-\lambda_i t_1}, ..., e^{-\lambda_i t_k})\ \forall i = 1, ..., n;\ \lambda_i \in \lambda(A)$. Since the eigenvalues of A are assumed to be rational, the denominator of λ_i is denoted by d_i $(i = 1, ..., n)$, and the least common multiple of denominators d_i's by $l = lcm(d_1, ..., d_n)$. Substituting $z_i = e^{-\frac{t_i}{l}}\ \forall i = 1, ..., k$, we can get parametric representations of x_i's in terms of z_i's. These representations are either polynomial or rational functions depending on the signs of the eigenvalues of A. It is evident that if the eigenvalues of A are either all positive or all negative, x_i's can be expressed as polynomials in z_i's. On the other hand, if the eigenvalues (recall that the eigenvalues are rational and non-zero) of A have mixed signs, we get rational parametric representations of x_i's.

Consider the case wherein some eigenvalues of A are positive and the remaining are negative: Rearrange the eigenvalues such that $\lambda_i, i = 1, ..., q$ are positive and $\lambda_j, j = q+1, ..., n$ (i.e. $n - q$ eigenvalues) are negative. On substituting $z_i = e^{-\frac{t_i}{l}}\ \forall i = 1, ..., k$, we get, $x_i = f_{ki}^+(z_1^{p_i}, ..., z_k^{p_i}), i = 1, ..., q$ and $x_i = f_{ki}^+(z_1^{-p_i}, ..., z_k^{-p_i}), i = q+1, ..., n$ where $p_i = |l\lambda_i|$ is an integer. While x_i's for $i = 1, ..., q$ are polynomials in $z_1, ..., z_k$, the x_i's for $i = q+1, ..., n$ can be written only as rational functions in the variables $z_1, ..., z_k$. Hence for both sets of x_i's defined above: $x_i = \frac{N_{ki}^+(z_1,...,z_k)}{D_{ki}^+(z_1,...,z_k)}, i = 1, ..., n$, where N_{ki}^+ and D_{ki}^+ are polynomial numerators and denominators of f_{ki} respectively and $D_{ki}^+ = 1\ \forall i = 1, ..., q$. This gives the following rational representation for M_k^+ and M_k^-:

$$M_k^+ = \left\{ (x_1, ..., x_n) : x_i = \frac{N_{ki}^+(z_1, ..., z_k)}{D_{ki}^+(z_1, ..., z_k)} \text{ and (4)} \right\} \quad (3)$$

$$0 < z_k \leq z_{k-1} \leq ... \leq z_1 \leq 1 \quad (4)$$

$$M_k^- = -M_k^+$$

Remark 6. Note that by substituting $z_i = e^{-\frac{t_i}{l}}$ (for the case where all eigenvalues of A are positive) or $z_i = e^{\frac{t_i}{l}}$ (for the case where all eigenvalues of A are positive), we have $D_{n-1,i}^+ = 1$ and $N_{n-1,i}^+ = f_{n-1,i}^+$ for all $i = 1, ..., n$ (which means that x_i's can be expressed as polynomials in z_i's).

The above expression for M_k^+ and M_k^- (i.e. (3) and (4)) involves $z_1, ..., z_k$ and hence cannot be used for state based switching. It would be convenient to eliminate the variables z_1, z_2,..., z_k from the representation of M_k^+ and M_k^- and find an alternate representation of M_k^+ and M_k^- in terms of only the state variables $x_1, ..., x_n$. Such a representation can be used directly for state-feedback based switching of the input values between ± 1. For this purpose, we will use an implicitization method based on the construction of Groebner bases to eliminate $z_1, ..., z_k$ and form an implicit representation for the set $M_k^{\pm}$. More details on Groebner basis methods can be found in [12], [14].

IV. Switching Surface and Feedback Logic

From the switching logic outlined at the end of section II, it seems that we need all the nested switching surfaces M_k, $k = 1, ..., n-1$ for implementing the switching feedback law. However, in practice, we will show that the lower dimensional switching surfaces (i.e. $M_k^{\pm}$ for $k < n-1$) are not essential for this purpose. Hence we only describe the implicitization procedure for $M_{n-1}^{\pm}$. If required, $M_k^{\pm}$, $k < n-1$ can be computed from $M_{n-1}^{\pm}$ easily (see remark 8).

A. Implicitization

1) Implicitization of (3): Parametric representations of $M_{n-1}^{\pm}$ is described by (3) for $k = n-1$. To eliminate z_k, $k = 1, ..., n-1$ from (3) we will follow the standard implicitization steps (please refer to [12]) described next:

a) Form an ideal $J_{n-1}^+ = \langle D_{n-1,1}^+ x_1 - N_{n-1,1}^+, ..., D_{n-1,n}^+ x_n - N_{n-1,n}^+, 1 - D_{n-1,1}^+ D_{n-1,2}^+ ... D_{n-1,n}^+ y \rangle$.

b) Compute Groebner basis G_{n-1}^+ of J_{n-1}^+ w.r.t. lexicographic ordering as $y \succ z_1 \succ z_2 \succ ... \succ z_{n-1} \succ x_1 \succ ... \succ x_n$.

c) The element $g_{n-1}^+ \in G_{n-1}^+ \cap \mathbb{Q}[x_1, ..., x_n]$ defines the smallest variety containing the parametric representation $x_i = f_{n-1,i}^+$. Similarly for M_{n-1}^-.

Remark 7. In general, the variety obtained by Groebner basis based elimination i.e. $\{(x_1, ..., x_n)$: $g_{n-1}^+(x_1, ..., x_n) = 0\}$, might be larger than $\left\{ (x_1, ..., x_n) : x_i = \frac{N_{n-1,i}^+(z_1, ..., z_{n-1})}{D_{n-1,i}^+(z_1, ..., z_{n-1})} \; i = 1, .., n \right\}$ and in particular, than M_{n-1}^+. However, by existence and uniqueness of time-optimal control, the set of solutions of the equation $g_{n-1}^+(x_1, ..., x_n) = 0$, for which there exist $z_1, ..., z_{n-1}$ satisfying the inequality (4) is exactly equal to M_{n-1}^+ i.e. $M_{n-1}^+ = \{(x_1, ..., x_n) : g_{n-1}^+(x_1, x_2, ..., x_n) = 0$, for $0 < z_{n-1}^+ \leq z_{n-2}^+ \leq ... \leq z_1^+ \leq 1\}$.

2) Implicitization of (4): For M_{n-1}^+, we denote the respective z_k by z_k^+ and similarly for M_{n-1}^-. To describe M_{n-1}^+ completely and ensure that $x_1, ..., x_n \in X_0$, we need to impose condition (4) in addition to $g_{n-1}^+ = 0$. For that we need to rewrite inequality (4) in terms of $x_1, x_2, ..., x_n$. This is accomplished by solving each z_k in terms of $x_1, x_2, ..., x_n$ and then imposing inequality (4). The technique described below for this purpose, works identically for each z_k^+, $k = 1, ..., n-1$. Let $G_{z_k}^+$ be the Groebner basis obtained by using ordering $.. \succ .. \succ ... \succ z_k^+ \succ x_1 \succ ... \succ x_n$. Consider the set $G_{z_k}^+ \cap \mathbb{Q}[z_k, x_1, ..., x_n]$: let there be m elements in this set and let us denote the elements by $g_1^+, g_2^+, ..., g_m^+$, each of which is a polynomial in the variables $(z_k, x_1, ..., x_n)$. Now, it might be possible to explicitly solve at least one among $g_i^+ = 0$ $(i = 1, ..., m)$, for z_k^+ (i.e. z_k^+ can be expressed as a closed form expression of $(x_1, .., x_n)$). This happens, for example, when in any one polynomial among g_i^+ $(i = 1, ..., m)$, all the monomials containing z_k^+ have the same degree of z_k^+. It turns out that such a situation occurs in all the numerical problems described in this paper.

However, in general, it is possible that all the polynomials $g_i^+ = 0$ $(i = 1, ..., m)$ contain monomials with varying degree in z_k^+ and consequently an explicit (closed form) solution for z_k^+ is not possible. In that case, whether the current state vector $(x_1, .., x_n)$ satisfies (4), can be checked based on the following real time computation. We substitute $(x_1, .., x_n)$ in to any one (usually the simplest) of $g_i^+ = 0$ $(i = 1, .., m)$, which then becomes a polynomial of the single variable z_k^+. This polynomial is then solved numerically for z_k^+. Finally we need to check whether any one set of solutions z_k^+ $(k = 1, ..., n-1)$ satisfies the corresponding inequality (4).

Remark 8. Though unnecessary in this scheme, it should be noted that all the lower dimensional switching surfaces M_{n-k} can be computed using M_{n-1} itself by $M_{n-k}^+ = \{(x_1, ..., x_n) : g_{n-1}^+(x_1, x_2, ..., x_n) = 0$, for $0 < z_{n-1}^+ \leq z_{n-2}^+ \leq ... \leq z_{k-1}^+ = ... = z_1^+ = 1\}$.

Remark 9. Further note that, the computationally intensive Groebner basis based implicitization is done off line, whereas the only on line computation required to implement the feedback switching logic involves *evaluation* of multivariate polynomials and in the worst case, solution of *univariate* polynomial equations.

B. Switching Algorithm

Using the above expression for M_{n-1}, any point on the state space $(x_1, ..., x_n)$ can be tested to check whether it belongs to the switching surface M_{n-1}. Now recall the first step of the switching logic described at the end of section II, which predicts that an optimal control for state trajectory starting from M_n should change its sign on intersecting M_{n-1}. Since we now have a description of M_{n-1}, this logic can now be implemented directly. However, an ideal implementation of the switching logic would require that the input should switch sign as soon as $x(t) \in M_{n-1}$. The trajectory, which is now *in* M_{n-1}, will travel along M_{n-1} until it intersects M_{n-2}. This situation should repeat for all the nested lower dimensional surfaces. Hence, clearly, for the ideal realization of optimal switching, we need to check $x(t) \in M_{n-1}$ instantaneously.

In any actual realization of the switching logic (either in simulation or in practice) due to unavoidable delay in computation (say δ seconds), the trajectory overshoots M_{n-1} at the first switch. The state-trajectory, which ideally should have stayed in M_{n-1} after hitting M_{n-1}, overshoots and enters M_n slightly before the first switch occurs. Since the state is now in M_n the switching surface M_{n-1} (instead of M_{n-2} required for ideal switching) can be used again for the second switch. This situation repeats for all successive switches and hence the entire switching law can be summarized as Algorithm 1 below. The initial input $u(0)$ can be determined by checking whether initial condition x_0 is in M_n^+ or M_n^- by using the algorithm described in section V. Moreover, it is evident that since M_{n-1} is a set of measure zero in M_n, generically, $x_0 \in M_n/M_{n-1}$.

Algorithm 1 Switching logic

Require: current $x(t)$ and $u(t)$
  if $x(t) \in M_{n-1}$ **then**
    $u(t+\delta) = -u(t)$
  else
    $u(t+\delta) = u(t)$
  end if

We need to guarantee that the trajectory generated by such delayed switching remains close to the actual time optimal trajectory and hence reaches arbitrarily close to the origin. This is shown below in lemma 10, for asymptotically decreasing switching delays.

Let a state trajectory start from an initial condition (x_0, t_0) with $x_0 \in \text{int}\,(X_0 \backslash M_{n-1})$ (This is generically true since M_{n-1} is a lower dimensional surface) and switch according to Algorithm 1, with delays $\delta_1, \delta_2, ..., \delta_n$ at the first n switchings. Let the corresponding switchings occur for state values $x_1, x_2, ..., x_n$ at the switching instants $t_1, t_2, ..., t_n$. We compare this trajectory to the corresponding optimal trajectory starting from x_0. Let the optimal trajectory switch at the values $x_1^{'}, x_2^{'}, ..., x_n^{'}$ at times $t_1^{'}, t_2^{'}, ..., t_n^{'}$. Clearly, $x_n^{'} = 0$. Because of the switching delays (say δ_i), the optimal trajectory should switch before the actual trajectory switches, such that $t_1 = t_1^{'} + \delta_1$. Denote the optimal trajectory by $x_o(t)$ and the trajectory generated by Algorithm 1 by $x_d(t)$. The following result holds:

Lemma 10. *For any* $x_0 \in$ *int* $(X_0 \backslash M_{n-1})$ *and any* $\epsilon > 0$, *there exist switching delays* $\delta_1, \delta_2, ..., \delta_n$ *with* $\delta_i > 0\ \forall i = 1, .., n$ *such that* $\|x_o(t) - x_d(t)\| < \epsilon\ \forall t \in [t_0, t_n]$ *and* $\|x_n\| \to 0$ *as* $\max(\delta_i) \to 0,\ i = 1, .., n$.

Proof: Let $x_0 \in \text{int}\,(M_n^+)$ (the following proof holds similarly for $x_0 \in \text{int}\,(M_n^-)$). Then, the first switch occurs for $x_o(t)$ when it reaches M_{n-1}^-, at $(x_1^{'}, t_1^{'})$, where $x_1^{'} \in M_{n-1}^-$. The $x_d(t)$, on the other hand, switches at (x_1, t_1) where $x_1 \in M_n^-$ and $t_1 = t_1^{'} + \delta_1$. Since $x(t)$ is a continuous function of t (see [5]), for any $\epsilon > 0$, $\exists \delta_1$ such that $\|x_1 - x_1^{'}\| < \epsilon$. The second portion of $x_o(t)$ starts from $(x_1^{'}, t_1^{'})$ and ends on M_{n-2}^+ at $(x_2^{'}, t_2^{'})$, where $x_2^{'} \in M_{n-2}^+$. The second portion of $x_d(t)$ starts from (x_1, t_1) on M_n^-, intersects M_{n-1}^+ at $(x_2^{''}, t_2^{''})$, overshoots due to the delay δ_2 and switches at (x_2, t_2), where $x_2 \in M_n^+$ and $t_2 = t_2^{''} + \delta_2$. Using similar arguments as above it is possible to choose δ_2 such that $x_2 \approx x_2^{''}$.

Next we claim that $x_2^{'} \approx x_2^{''}$. To prove this we consider two time optimal trajectories starting from x_1 and $x_1^{'}$ respectively. Since $x_1 \in M_n^-$ and $x_1^{'} \in M_{n-1}^-$, the first portions of both these trajectories use the same initial input $u = -1$ and these trajectories will first switch at $(x_2^{''}, t_2^{''})$ and $(x_2^{'}, t_2^{'})$ respectively. Evidently, the switching instants for these two trajectories will be the corresponding solutions of $x_1 = F_n^-$ and $x_1^{'} = F_n^-$, which can be converted in to (see section IV-A2 and V) polynomials in $(x_1; z_2^{''})$ and $(x_1^{'}; z_2^{'})$ respectively (where $z_2^{''}$ and $z_2^{'}$ are the transformed $t_2^{''}$ and $t_2^{'}$ respectively). Now, it is well known that roots of univariate polynomials are continuous functions of its coefficients. Hence, since $x_1 \approx x_1^{'}$, it follows that $z_2^{''} \approx z_2^{'}$, $t_2^{'} \approx t_2^{''}$ and $(t_2^{'} - t_1^{'}) \approx (t_2^{''} - t_1)$. Then from continuity of the vector field (corresponding to $u = -1$), it follows that $x_2^{'} \approx x_2^{''}$. Hence, there exists δ_2 such that $x_2 \approx x_2^{'}$. This argument extends to subsequent switching instants $i = 2, ..., n$, showing that there exists $\delta_1, \delta_2, ..., \delta_i$ such that $x_i \approx x_i^{'}$ for all $i = 1, ..., n$. Therefore, we conclude that for any $\epsilon > 0$, there exist switching delays $\delta_1, \delta_2, ..., \delta_n$ with $\delta_i > 0\ \forall i = 1, .., n$ such that $\|x_o(t) - x_d(t)\| < \epsilon$ $\forall t \in [t_0, t_n]$. Moreover, as $\max(\delta_i) \to 0$, we have $x_i \to x_i^{'}$ for all $i = 1, ..., n$. In particular $x_n \to x_n^{'}$. But, we have $x_n^{'} = 0$. Thus $\|x_n\| \to 0$ as $\max(\delta_i) \to 0$ for all $i = 1, .., n$. ■

Remark 11. In practical implementation with a fixed sampling frequency, the trajectory may intersect M_{n-1} in between two sampling instants, whereas at the sampling instants $g_{n-1}^{\pm} \neq 0$. However, by continuity of the (real parts of) solutions of polynomial equations, the corresponding inequalities (4) are valid in small neighborhoods of M_{n-1}. Hence these inequalities are checked first at each sampling instant, and once any one set of inequalities are valid, $g_{n-1}^{\pm}$ is checked for changes of sign between two consecutive sampling instants. A change of sign indicates that the trajectory has just crossed M_{n-1}.

C. Examples of (Near) Time Optimal Feedback

Example 12. Consider a system defined as (1) with $A = \text{diag}(1, 2, 3)$ and $B = [1\ 1\ 1]^T$. The points on the switching surface are described by M_2^+ and M_2^-. The equations $x = F_2^+(z_1, z_2)$ and F_2^- are calculated using equation (3) and (4). The polynomial equations $x = F_2^+$, along with the inequalities $0 < z_2 \leq z_1 \leq 1$ describe the surface M_2^+ (see (3) and (4)). To eliminate z_1 and z_2 from the description of M_2^+ we follow the implicitization procedure described in section IV. We form the following ideal from the polynomials in F_2^+: $J = \langle x_1 - 2z_1 + z_2 + 1, x_2 - z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}, x_3 - \frac{2}{3}z_1^3 + \frac{1}{3}z_2^3 + \frac{1}{3} \rangle$. A successful elimination of z_1 and z_2 would give an equation of the form $g_2^+(x_1, x_2, x_3) = 0$. For this purpose we calculate the Groebner basis G of the ideal J with the ordering as $z_1 \succ z_2 \succ x_1 \succ x_2 \succ x_3$. Only relevant elements of the basis are

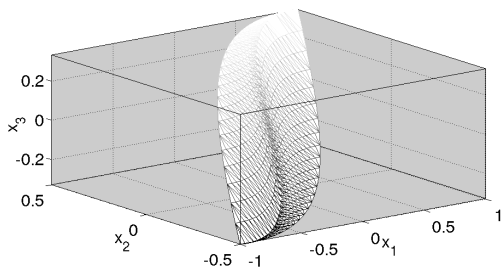

Fig. 2. Points on the Switching Surface (Example 12)

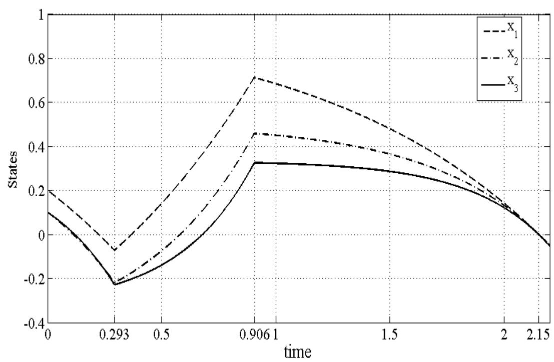

Fig. 3. State against time for $x_0 = (0.2, 0.1, 0.1)$ (Example 12)

shown below:

$$\begin{aligned} s_1 &= x_1^6 + 6x_1^5 - 6x_1^4x_2 + 12x_1^4 - 24x_1^3x_2 - 24x_1^3x_3 + \\ &36x_1^2x_2^2 - 72x_1^2x_3 - 18x_1^2 + 72x_1x_2^2 + 72x_1x_2x_3 \\ &+ 72x_1x_2 - 36x_1x_3 - 72x_2^3 - 72x_2^2 + 72x_2x_3 - 18x_3^2 \\ s_5 &= 3z_2(x_1^2 + 6x_1 - 6x_2) + x_1^3 + 3x_1^2 - 6x_1x_2 - 6x_2 + 6x_3 \end{aligned} \quad (5)$$

Clearly s_1 does not involve z_1 or z_2; thus $g_2^+(x_1, x_2, x_3) = s_1$. Along with conditions $0 < z_2 \leq z_1 \leq 1$ on z_1 and z_2, $g_2^+(x_1, x_2, x_3) = 0$ describes M_2^+. We observe s_5 contains only degree one terms in z_2. Thus, we can write an expression for z_2 as, $z_2 = \frac{-(x_1^3+3x_1^2-6x_1x_2-6x_2+6x_3)}{(3x_1^2+6x_1-6x_2)}$. Similar expression for z_1 can be found from the Groebner basis of the ideal J with ordering $z_2 \succ z_1 \succ x_1 \succ x_2 \succ x_3$, which, after similar calculations turn out to be $z_1 = \frac{-(-x_1^3-3x_1^2-3x_1+3x_3)}{(3x_1^2+6x_1-6x_2)}$. Therefore, $M_2^+ = \{(x_1, x_2, x_3) : s_1 = 0, \text{ for } 0 < z_2 \leq z_1 \leq 1\}$. One can obtain M_2^- just by replacing (x_1, x_2, x_3) as $(-x_1, -x_2, -x_3)$ in M_2^+. Figure 2 shows points on M_2^+ and M_2^-. The feedback logic can be implemented according to algorithm 1

Next we compare, for a sample initial condition, the trajectories generated according to algorithm 1, with those generated by optimal open loop input computed according to PMP. For initial condition $x_0 = \begin{bmatrix} 0.2 & 0.1 & 0.1 \end{bmatrix}$ we have $u_0 = -1$. The respective calculated open-loop switching instants (according to PMP) are $t_1 = 0.293$, $t_2 = 0.906$, and at $t_3 = 2.15$ the system states reach the origin. We implement the proposed algorithm 1 for this value of x_0. It can be observed from figure 3, that the open-loop switching instants and closed loop switching instants are matching.

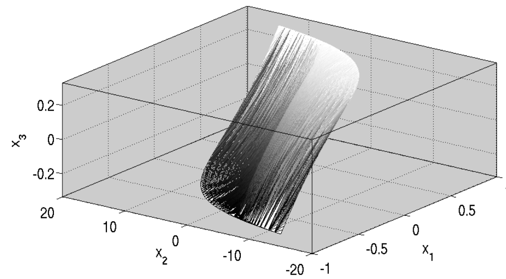

Fig. 4. Points on the Switching Surface (Example 13)

Example 13. Next we consider a similar system to the examples above except that, $A = \text{diag}(1, -2, 3)$. Figure 4 shows the switching surface. Note that the switching surface is bounded in the unstable directions: x_1 and x_3, whereas it is unbounded in the stable direction x_2. We have arbitrarily truncated the switching surface in the x_2 direction between $-12 \leq x_2 \leq 12$, in figure 4 for ease of representation.

V. Region of Null-controllability (X_0)

According to the discussion from section II, $X_0 = M_n^+ \cup M_n^-$. Recall from section III, the different representations of M_n according to the sign of the eigenvalues of A. It was shown in [22], [23] that if $A = diag(A_1, A_2)$, with $A_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ is with positive eigenvalues and $A_2 \in \mathbb{R}^{m\times m}$ is with negative eigenvalues and B is partitioned accordingly as $B = \begin{bmatrix} B_1^T & B_2^T \end{bmatrix}^T$ then, $X_0 = X_0' \times \mathbb{R}^m$ where X_0' is the null-controllable region of the system $\dot{x}_1 = A_1x_1 + B_1u$. Thus, it is enough to consider only the case in which all eigenvalues of A are positive.

Given any arbitrary $x \in \mathbb{R}^n$, we would like to check whether $x \in X_0$. We know, $M_n^+ = \{x_i = f_{ni}^+(z_1^{p_i}, ..., z_n^{p_i}) \forall i = 1, ..., n : 0 < z_n \leq z_{n-1} \leq ... \leq z_1 \leq 1\}$. Since each x_i is expressed as a polynomial in z_i, $i = 1, ..., n$, we can systematically eliminate z_i's by using the Groebner basis based elimination technique. We first form an ideal $J_n^+ = \langle x_1 - f_{n1}^+, ..., x_n - f_{nn}^+ \rangle \subset \mathbb{Q}[z_1, ..., z_n, x_1, ..., x_n]$, and by using lexicographic ordering $z_1 \succ z_2 \succ ... \succ z_n \succ x_1 \succ ... \succ x_n$, compute Groebner basis G_n^+ for J_n^+. Then select the element in G_n^+ which contains none of the variables $z_1, z_2, ..., z_{n-1}$ i.e. $g_n^+ = G_n^+ \cap \mathbb{Q}[x_1, ..., x_n, z_n]$ and solve for z_n. Since we know the test point x, the solution of $g_n^+ = 0$ for z_n involves, solving a univariate polynomial equation. Now, select an element in G_n^+ which contains none of the variables $z_1, z_2, ..., z_{n-2}$ i.e. $g_{n-1}^+ \in G_n^+ \cap \mathbb{Q}[x_1, ..., x_n, z_n, z_{n-1}]$. Using the values of z_n obtained in the previous step solve for z_{n-1} and so on. Finally, select an element in G_n^+ which contains all variables $z_1, ..., z_n$ i.e. $g_1^+ \in G_n^+ \cap \mathbb{Q}[x_1, ..., x_n, z_n, z_{n-1}, ..., z_1]$. Using the values of $z_n, z_{n-1}, ..., z_2$ obtained in the previous steps solve for z_1. Now, if there exist any one combination of $z_n, z_{n-1}, ...z_1$ satisfying the inequality $0 < z_n \leq z_{n-1} \leq ... \leq z_1 \leq 1$ then $x \in M_n^+$. A similar procedure holds for M_n^-. However, the above procedure involves solving more than n univariate polynomial equations in $z_1, ..., z_n$ and verification of n-inequalities. This can be numerically difficult for large system orders. Hence we propose a simpler sufficient condition to test whether $x \in X_0$.

Consider $g_n^+(x_1,...,x_n,z_n) = G_n^+ \cap \mathbb{Q}[x_1,...,x_n,z_n]$ defined above. By the very nature of the parametrization, $t_n \to \infty$ implies $z_n = e^{\frac{-t_n}{l}} \to 0$ (where $l = lcm(d_1,d_2,...,d_n)$ is the least common multiple of the numbers $d_1,d_2,...,d_n$). Substituting $z_n = 0$ in the equation $g_n^+ = 0$ identifies exactly those $(x_1,...,x_n)$ from where it takes infinite time to reach origin. Hence by putting $z_n = 0$, we can select a surface which contains the boundary of set M_n^+. A similar procedure holds for M_n^-. Hence the sets $\{(x_1,..,x_n) \in \mathbb{R}^n : g_n^+(x_1,...,x_n,0) = 0\}$ and $\{(x_1,...,x_n) \in \mathbb{R}^n : g_n^-(x_1,...,x_n,0) = 0\}$ together contains the boundary of null-controllable region. The following theorem 14 uses these implicit expressions for the boundary of X_0 and gives a sufficient condition to check whether a given point $x \in X_0$:

Theorem 14. *If for all $\mu \in [0,1]$, $g_n^+(\mu x) \neq 0$ and $g_n^-(\mu x) \neq 0$ then $x \in X_0$.*

Proof: By lemma 3, $X_0 = M_n$. Also, $0 \in M_n$ and $M_n = M_n^+ \cup M_n^-$. The boundary of M_n^+ is contained in the set $\{(x_1,..,x_n) \in \mathbb{R}^n : g_n^+(x_1,...,x_n,0) = 0\}$ and similarly boundary of M_n^- is contained in $\{(x_1,..,x_n) \in \mathbb{R}^n : g_n^-(x_1,...,x_n,0) = 0\}$. Now, for all $x \in X_0$, the line segment joining x and origin should not intersect boundary of X_0. Thus, for all $\mu \in [0,1]$, if $g_n^+(\mu x) \neq 0$ and $g_n^-(\mu x) \neq 0$ then $x \in X_0$. ■

However, this condition is only sufficient since parts of the surfaces defined by $g_n^+ = 0$ and $g_n^- = 0$ can lie in the interior of X_0. Then either $g_n^+(\mu x) = 0$ or $g_n^-(\mu x) = 0$ can hold for some μ even if a particular $x \in X_0$. In such a situation, we check whether $\mu x \in X_0$ by using the inequalities described above.

Example 15. We use the system from example 12. The set X_0 is divided into two parts: M_3^+ and M_3^-. The functions F_3^+ and F_3^- are computed using (2) and converted into polynomials using (3). Next we form an ideal $I = \langle 2z_1 - 2z_2 + z_3 - x_1 - 1, z_1^2 - z_2^2 + \frac{1}{2}z_3^2 - x_2 - \frac{1}{2}, \frac{2}{3}z_1^3 - \frac{2}{3}z_2^3 + \frac{1}{3}z_3^3 - x_3 - \frac{1}{3}\rangle$. We compute Groebner basis G of ideal I w.r.t. lexicographic ordering $z_1 \succ z_2 \succ z_3 \succ x_1 \succ x_2 \succ x_3$. We select an element in $G \cap \mathbb{Q}[z_3,x_1,x_2,x_3]$ which is as follows: $s = 3z_3^4 - 12z_3^3x_1 - 12z_3^3 - 6z_3^2x_1^2 - 12z_3^2x_1 + 48z_3^2x_2 + 18z_3^2 + 4z_3x_1^3 + 12z_3x_1^2 + 12z_3x_1 - 48z_3x_3 - 4z_3 - x_1^4 - 4x_1^3 - 2x_1^2 + 48x_1x_3 + 12x_1 - 48x_2^2 - 48x_2 + 48x_3 + 1$. Thus $g_3^+ = s$ and the set of points satisfying $g_3^+(0,x_1,x_2,x_3) = 0$ and $g_3^+(0,-x_1,-x_2,-x_3) = 0$ contains the boundary of set X_0. For this example $-x_1^4 - 4x_1^3 - 6x_1^2 + 48x_1x_3 + 12x_1 - 48x_2^2 - 48x_2 + 48x_3 + 3 = 0$ and $-x_1^4 + 4x_1^3 - 6x_1^2 + 48x_1x_3 - 12x_1 - 48x_2^2 + 48x_2 - 48x_3 + 3 = 0$ form the boundary of X_0.

Next we check whether the point $x = [0.2, 0.1, 0.1]$ lies in X_0 by applying theorem 14. For a given x, we get $g_3^+(\mu x) = \mu^4 - 20\mu^3 - 150\mu^2 + 1500\mu - 1875$. Roots of this polynomial are $-23.79, 10.32, -5.00$ and -1.527. Similarly the roots of the polynomial $g_3^-(\mu x) = g_3^+(-\mu x)$, are $23.79, -10.32, 5.00$ and 1.527. None of these roots lie in the interval $[0,1]$ which proves that $x \in X_0$.

VI. Conclusions

In this article we use a Groebner basis based implicitization technique to synthesize a feedback based switching law that nearly produces time-optimal switching. This switching feedback law robustly drives the system to an arbitrarily small neighborhood of origin in finite time. The proposed method is currently limited to controllable diagonalizable systems with non-zero rational eigenvalues. Relaxing some of these assumptions is the subject of current and future research.

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