# Department of Mathematics
# MTL 725(Stochastic Processes and Applications))
# Tutorial Sheet No. 2

1. The one-step transition probability matrix of a discrete time Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ having three states 1, 2 and 3 is

$$P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

   and the initial distribution is

$$\Pi(0) = (0.7, 0.2, 0.1)$$

   Find

   (a) $P(X_2 = 3)$

   (b) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

2. Let $\{X_n, n = 0, 1, 2, \ldots\}$ be a Markov chain with state space $\{0, 1, 2\}$, the initial probability vector $\Pi(0) = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ and one step transition probability matrix $P = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$

   (a) Compute $P(X_0 = 0, X_1 = 1, X_2 = 1), P(X_2 = 1)$

   (b) Compute $P(X_1 = 1, X_2 = 1/X_0 = 0)$

   (c) Compute $P(X_4 = 1/X_2 = 2), P(X_7 = 0/X_5 = 0)$.

3. Consider a communication system which transmits the two digits 0 or 1 through several stages. Let $X_0$ be the digit transmitted initially (leaving) $0th$ stage and $X_n, n \geq 1$ be the digit leaving the $nth$ stage. At each stage there is a constant probability $q$ that the digit which enters is transmitted unchanged and probability $p$ otherwise. $p+q = 1$. Show that $\{X_n, n = 0, 1, \ldots\}$ is a Markov chain. Find its one step transition probability matrix $P$ and compute $P^m$. Find $lim_{m\to\infty} P^m$. Also compute $P(X_0 = 0/X_m = 0)$ and $P(X_m = 0)$.

4. A factory has two machines and one repair crew. Assume that probability of any one machine breaking down on a given day is $\alpha$. Assume that if the repair crew is working on a machine, the probability that they will complete the repairs in one more day is $\beta$. For simplicity, ignore the probability of a repair completion or a breakdown taking place except at the end of a day. Let $X_n$ be the number of machines in operation at the end of the $nth$ day. Assume that the behavior of $X_n$ can be modeled as a Markov chain. $(a)$ Find one step transition probability matrix for the chain.
   (b) If the system starts out with both the machine operating, what is the probability that both will be in operation two day later?

5. Show that if a Markov chain is irreducible and $P_{ii} > 0$ for some state $i$ then the chain is aperiodic.

6. Consider a DTMC with states 0, 1, 2, 3, 4. Suppose $p_{0,4} = 1$; and suppose that when the chain is in state $i$, $i > 0$, the next state is equally likely to be any of the states $0, 1, \ldots, i-1$.

   (a) Discuss the nature of the states of this Markov chain.

   (b) Discuss whether there exist a limiting distribution and find one if it exists.

7. Let 0 be an absorbing state and for $j > 0$, $P_{jj} = p$, $P_{jj-1} = q$ where $p + q = 1$. Find $f_{j0}^{(n)}$, the probability that absorption takes place exactly at $nth$ step given initial state is $j$. Also find the expectation of this distribution.

8. Consider a time homogeneous discrete time Markov chain with one step transition probability matrix $P$ and states $\{0, 1, 2, 3, 4\}$ where

$$P = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

Classify the states of the Markov chain as positive recurrent, null recurrent or transient. Discuss the behaviour of $P_{ij}^{(n)}$ as $n \to \infty$ for all $i, j = 0, 1, \ldots, 4$.

9. Show that the DTMC with countable state space $S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and transition probabilities

$$p_{i,i+1} = p_{i,i-1}, \quad i = 0, \pm 1, \pm 2, \ldots$$

is recurrent when $p = 1/2$ and transient when $p \neq 1/2$.

10. Consider a gambler who at each play of the game has probability $p$ of winning one unit and probability $q = 1 - p$ of losing one unit. Assume that successive plays of the game are independent. Suppose the gambler's fortune is presently $i$, and suppose that we know that the gambler's fortune will eventually reach $N$ (before it goes to 0). Given this information, show that the probability he wins the next game is

$$\begin{array}{ll} \frac{p[1-(q/p)^{i+1}]}{1-(q/p)^i}, & \text{if } p \neq \frac{1}{2} \\ \\ \frac{i+1}{2i}, & \text{if } p = \frac{1}{2} \end{array}$$

11. Let $\{Y_n, n = 1, 2, \ldots\}$ be a sequence of independent random variables with

$$P(Y_n = 1) = p = 1 - P(Y_n = -1).$$

Let $X_n$ be defined by $X_0 = 0, \quad X_{n+1} = X_n + Y_{n+1}$ Examine whether $\{X_n, n = 1, 2, \ldots\}$ is a Markov chain. Find $P\{X_n = k\}, k = 0, 1, 2, \ldots$

12. Show that, for a chain with a finite number, $m$ of states and having a doubly stochastic matrix $(P_{jk})$ for its transition matrix,

$$P_{jk}^{(n)} \to v_k = 1/m, \quad \text{for all } \; j, k = 1, 2, \ldots, m.$$

(A non-negative square matrix is said to be doubly stochastic matrix if all the row and column sums are unity.)

13. The owner of a local one-chair barber shop is thinking of expanding because there seem to be too many people waiting. Observations indicate that in the time required to cut one person's hair there may be 0,1 and 2 arrivals with probability .3,.4 and .3 respectively. The shop has a fixed capacity of six people whose hair is being cut. Let $X_n$ be the number of people in the shop (excluding the barber and $nth$ person) at the completion of the $nth$ person's hair cut. $\{X_n, n = 1, 2, \ldots\}$ is a Markov chain assuming i.i.d arrivals. Find its one step transition probability matrix. Determine the 'long run' proportion of time that the shop has six people in it; that it has 5 people in it.

14. For a Markov chain $\{X_n, n = 1, 2, \ldots\}$ with state space $S = \{0, 1, 2, 3, 4\}$ and transition probability matrix $P$ given below, classify the states of the chain. Also determine the closed communicating classes.

$$(a) \quad P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \qquad (b) \quad P = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

15. Discuss the limiting behavior as $n \to \infty$ of Markov chain with $P$ given in 14(a).