Department of Mathematics
MTL 725(Stochastic Processes and Applications))
Answers to Tutorial Sheet No. 2

1. (a) $P(X_2 = 3) = 0.279$
(b) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2) = 0.0048$

2. (a) $\frac{1}{16}$ (b) $\frac{1}{4}$ (c) $\frac{13}{48}$ $\frac{5}{16}$

3. $(i) P = \begin{pmatrix} q & p \\ p & q \end{pmatrix}$

(ii) $P^m = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(q-p)^m & \frac{1}{2} - \frac{1}{2}(q-p)^m \\ \frac{1}{2} - \frac{1}{2}(q-p)^m & \frac{1}{2} + \frac{1}{2}(q-p)^m \end{pmatrix}$

(iii) $\lim_{n\to\infty} P^n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

(iv) $P(X_0 = 0 | X_m = 0) = \frac{P(X_m=0|X_0=0)P(X_0=0)}{P(X_m=0)} = \frac{1}{2} - \frac{1}{2}(q-p)^m$

(v) $P(X_m = 0) = P(X_m = 0|X_0 = 0)P(X_0 = 0) + P(X_m = 0|X_0 = 1)P(X_0 = 1) = \frac{1}{2}$

Assume $P(X_0 = 0) = \frac{1}{2}$ and $P(X_0 = 1) = \frac{1}{2}$

4. (a) $\begin{pmatrix} (1-\beta) & \beta & 0 \\ \alpha(1-\beta) & (1-\alpha)(1-\beta)+\alpha\beta & \beta(1-\alpha) \\ \alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2 \end{pmatrix}$ (b) $P_{22}^{(2)}$

6. (a) The Chain is irreducible, aperiodic and all states are positive recurrent.
(b) $(v_0, v_1, v_2, v_3, v_4) = (\frac{12}{37}, \frac{6}{37}, \frac{4}{37}, \frac{3}{37}, \frac{12}{37})$

7. $f_{j0}^{(n)} = \begin{cases} {}^{n-1}C_j q^j p^{n-j} & \text{if } j \le n \\ 0 & \text{if } j > n \end{cases}$

$E[f_{j0}^{(n)}] = j\left[q^j + p/q\right]$

8. (a) $C_1 = \{0, 2\}$ and $C_2 = \{3, 4\}$ are two communicating classes. Both are aperiodic. These states are positive recurrent. $\{1\}$ is a transient state.
(b) $\lim_{n\to\infty} p_{ij}^{(n)}$ will exist depending on the initial state. It will be unique since the two chains are positive recurrent and aperiodic.
If initial state = 0, 1 or 2 then $(v_0, v_1, v_2, v_3, v_4) = (\frac{1}{2}, 0, \frac{1}{2}, 0, 0)$
If Initial state = 3 or 4 then $(v_0, v_1, v_2, v_3, v_4) = (0, 0, 0, \frac{1}{2}, \frac{1}{2})$

11. $P(X_n = k) = \begin{cases} {}^nC_{(n+k)/2}\ p^{(n+k)/2} q^{(n-k)/2} & \text{if } k \in 2\mathbb{Z}^+ \text{ and } n \text{ is even} \\ & k \in \mathbb{Z}^+ - 2\mathbb{Z}^+ \text{ and } n \text{ is odd} \\ {}^nC_{(n-k)/2}\ p^{(n-k)/2} q^{(n+k)/2} & \text{if } k \in 2\mathbb{Z}^- \text{ and } n \text{ is even} \\ & k \in \mathbb{Z}^+ - 2\mathbb{Z}^+ \text{ and } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$

13. State Space={0,1,2,3,4,5}

$$\text{One-Step Transition Probability Matrix} = P = \begin{pmatrix} 0.3 & 0.4 & 0.3 & 0 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.4 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.4 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0.3 & 0.7 \end{pmatrix}$$

Long run proportion of time that the shop will have 6 people in it $= v_5$

14. (a) {3}- absorbing state, {0, 1, 2, 4} - transient states, {3} - closed communicating class
(b) {2, 4}-Communicating class, {0, 1, 3}-transient states