Department of Mathematics
MTL 725(Stochastic Processes and Applications))
Tutorial Sheet No. 4

1. Let $\{X(t), t \geq 0\}$ be the Brownian motion. Prove that $(X(t_1), X(t_2), \ldots, X(t_n))$ is jointly normal distributed. Also find the mean and variance for a r.v. $X(t)$.

2. Let $S(t)$ be the stock price at any time t. Assume that $\{S(t), t \geq 0\}$ follows a geometric Brownian motion. Prove that $E(S(t)) = S(0)e^{t(\mu+\frac{\sigma^2}{2})}$ where $S(0)$ is the initial price, μ is the drift parameter and σ is the volatility parameter.

3. Let X be a normally distributed random variable with mean μ and variance σ^2. Let u be a fixed number in $\mathbf{R}$ and define the convex function $\phi(x) = e^{ux}$ for all $x \in \mathbf{R}$. Prove that (a) $E(\phi(X)) = e^{u\mu+\frac{1}{2}u^2\sigma^2}$(b) Verify the Jensen's inequality holds $E(\phi(X)) \geq \phi(E(X))$.

4. Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with $P(X_1 = 1) = p$ and $P(X_1 = -1) = q = 1 - p, 0 < p < 1$. Define $Z_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$ and $Z_o = 0$. Show that Z_n and X_{n+1} are independent random variable.

5. Show that for any $T > 0$,
$$V(t) = W(t+T) - W(T)$$
is a Wiener process if $W(t)$ is a Wiener process.

6. Let $\{W(t), t \geq 0\}$ be a Brownian motion. Prove or disprove that $\{tW(1/t), t \geq 0\}$ where $tW(1/t)$ is taken to be zero when $t = 0$, is a Brownian motion.

7. Prove that, the solution of diffusion equation
$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}; \quad f = f(x,t)$$
is the probability density function of normal distribution $N(0,t)$.

8. Prove that $S(t) = e^{-\lambda\sigma t}(\sigma+1)^{N(t)}$, $\sigma > -1$ is a constant, is a martingale.

9. Let $\{W(t),\ t \geq 0\}$ be a Brownian motion and let $\{N(t),\ t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$, both defined on the same probability space (Ω, F, P) and relative to the same filtration $F(t)$, $t \geq 0$. Prove that $W(t)$ and N(t) are independent.

10. Let $\{Q(t),\ t \geq 0\}$ be a compound Poisson process and let $0 = t_0 < t_1 < \ldots < t_n$. Prove that the increments $Q(t_1) - Q(t_0)$, $Q(t_2) - Q(t_1)$, $\ldots$, $Q(t_n) - Q(t_{n-1})$ are stationary and independent. Also prove that the distribution of $Q(t_j) - Q(t_{j-1})$ is the same as the distribution of $Q(t_j - t_{j-1})$.

11. Let $\{W(t), t \geq 0\}$ be a Wiener process. Find the conditional distribution of $W(t_1/2)$ given that $W(t_1) = x_1$.

12. Let S_k be the price of a risky asset at time $k = 0, 1, 2, \ldots, n$.

    Let $S_{k+1} = \begin{cases} uS_k, & \text{with prob p;} \\ dS_k, & \text{with prob 1-p.} \end{cases}$

    Prove that $E(\ln S_n/S_{n-1}, S_{n-2}, \ldots, S_0) = \ln S_{n-1} + p\ln u + (1-p)\ln d$. Define a related process R_k as
$$R_k = \ln S_k - k[p\ln u + (1-p)\ln d].$$
    Prove that R_n is a martingale.

13. Let $\{X_n, n = 0, 1, 2, \ldots\}$, $\{Y_n, n = 0, 1, 2, \ldots\}$ be stochastic processes. We say $\{X_n\}$ is a martingale with respect to $\{Y_n\}$ iff
$$(a)\ \ E[|X_n|] < \infty \text{ and } (b)\ \ E[X_{n+1}/Y_0, Y_1, Y_2, \ldots, Y_n] = X_n.$$
    Prove that $\{X_n\}$ is a martingale with respect to $\{Y_n\}$ where $X_n = Y_1 + Y_2 + \ldots + Y_n$, $n \geq 1$, $Y_0 = 0$, $\{Y_i, i = 1, 2, \ldots\}$ are independent random variable with $E(Y_n) = 0$.

14. Let X_n be a symmetric random walk and $\mathbb{F}_n$ be a filtration. Show that

$$Y_n = (-1)^n cos(\pi X_n)$$

is a martingale with respect to $\mathbb{F}_n$.

15. Let the interest rate r and the volatility $\sigma > 0$ be constant. Let $S(t)$ be the stock price at any time t. Let

$$S(t) = S(0)e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W(t)}$$

be a geometric Brownian motion with mean rate of return r, where the initial stock price $S(0)$ is positive and $W(t)$ is a Brownian motion. Let K be a positive constant. Show that, for $T > 0$,

$$E\left[e^{-rT}(S(T) - K)^+\right] = S(0)N(d_+(T, S(0))) - Ke^{-rT}N(d_-(T, S(0))),$$

where $X^+ = Max(X, 0)$,

$$d_\pm(T, S(0)) = \frac{1}{\sigma\sqrt{T}}\left[log\frac{S(0)}{K} + \left(r \pm \frac{\sigma^2}{2}\right)T\right],$$

and N is the cumulative standard normal distribution function

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}z^2}dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{1}{2}x^2}dx.$$

16. Let X_n be a sequence of square integrable random variables. Show that if X_n is a martingale with respect to a filtration $\mathbb{F}_n$, then X_n^2 is a sub martingale with respect to the same filtration. (Hint: Use Jensen's inequality with convex function $\phi(x) = x^2$)

17. Prove that $\{W(t)^2 - t, t \geq 0\}$ is a martingale, where $\{W(t), t \geq 0\}$ is a brownian motion.

18. Let $\{W(t), t \geq 0\}$ be a Wiener process. Find the conditional distribution of $W(t)$ given that $W(s) = c$ (where c is a constant) when $s < t$. Is $exp(\sigma W(t) - \frac{\sigma^2}{2}t)$ a martingale where σ is a positive constant? Justify your answer.

19. Let $\{X(t), t \geq 0\}$ be a Poission process with rate 1. Which of the following are martingales.

    (a) $\{X(t) - t, t \geq 0\}$

    (b) $\{X(t)^2 - t, t \geq 0\}$

    (c) $\{(X(t) - t)^2 - t, t \geq 0\}$

    Justify your answers.