Department of Mathematics
MTL 725(Stochastic Processes and Applications))
Tutorial Sheet No. 5

1. Let $\{N(t) : t \geq 0\}$ be a Poisson process.

   (a) Find the distribution of current life, δ_t and total life β_t

   (b) Prove that the joint distribution of ν_t and δ_t is given by
   $P(\nu_t > x, \delta_t > y) = \begin{cases} e^{-\lambda(x+y)}, & \text{if } x > 0, 0 < y < t \\ 0, \text{if } y \geq t \end{cases}$

2. Show that the renewal function corresponding to the lifetime density $f(x) = \lambda^2 x e^{-\lambda x}, x \geq 0$ is $M(t) = \frac{\lambda t}{2} - \frac{1-e^{-2\lambda t}}{4}$

3. Let $\{N(t) : t \geq 0\}$ be a renewal process. Prove that $\lim_{t\to\infty} \frac{V(t)}{t} = \frac{\sigma^2}{\mu^3}$ where $V(t)$ is the variance of $N(t)$, μ and $\sigma^2 < \infty$ are the mean and variance, respectively, of the inter-arrival distribution.

4. If the arrivals in the queue follow a Poisson process, then the successive times X_k from the commencement of the of the k^{th} busy period to the start of the next busy period form a renewal process. (A busy period is an uninterrupted duration when the queue is not empty). Each X_k is composed of a busy portion Z_k and an idle portion Y_k. Prove that, $P(t)$, the probability that the queue is empty at time t converges to $E(Y_1)/E(X_1)$.

5. Let $X_1,\ X_2, \ldots$ be the interoccurence times in a renewal process. Suppose $P(X_i = 1) = \frac{1}{3},\ P(X_i = 2) = \frac{2}{3},\ i = 1, 2, \ldots.$ Let N_n be the number of renewals upto discrete time n. Compute $P(N(1) = k),\ P(N(2) = k),\ P(N(3) = k)$.

6. At the beginning of each day customers arrive at a taxi stand a t times of a renewal process with distribution law $F(x)$. Assume an unlimited supply of cabs as at an airport. Suppose each customer pays a random fee at the station following the distribution law $G(x),\ x > 0$.

   (i) Write the expression for the sum of money collected by the station, by time t of the day.

   (ii) Find $\lim_{t\to\infty} E(\text{the money collected over an initial interval of time t})/t$

7. Find the steady state and time dependent probabilities of $G/M/1/2$ and $M/G/1/2$. Assume that G follows Erlang distribution with parameters 2 and 1/3. Also, assume that, for $G/M/1/2$, $\mu = 2$ and for $M/G/1/2$, $\lambda = 2$.

8. Consider a $M/E_k/1$ queueing system with arrival rate λ and Mean service time is $\frac{1}{\mu}$. Show that the probability Generating function of the distribution of system state in Equilibrium is given by

$$V(s) = \frac{(1-\rho)(1-s)}{1 - s\left[1 + \frac{\rho(1-s)}{k}\right]^k}.$$

9. In $M/G/1$ queueing system, Suppose μ_{ij} is the mean First passage time of state j starting from state I. Prove that

$$\mu_{ij} = \mu_I + \sum_{k\neq j} P_i \mu_{kj}$$

   where μ_i is the mean sojurn time in state i.