Department of Mathematics
MTL 390 (Statistical Methods)
Tutorial Sheet No. 1

1. Determine the sample space for each of the following random experiments.
   (a) A student is selected at random from statistics lecture class and the students total marks is determined.
   (b) A coin is tossed three times, and the sequence of heads and tails is observed.

2. One urn contains three red balls, two white balls and one blue ball. A second urn contains one red ball, two white balls and three blue balls:
   (a) One ball is selected at random from each urn. Describe the sample space.
   (b) If the balls in two urns are mixed in a single urn and then a sample of three is drawn, find the probability that all three colors are represented when sample is selected
      (i) with replacement
      (ii) without replacement.

3. A fair coin is continuously flipped. What is the probability that the first five flips are
   (a) H, T, H, T, T
   (b) T, H, H, T, H.

4. The first generation of particle is coefficient of offsprings of a given particle. The next generation is formed by the offsprings of these members. If the probability that a particle has k offsprings (split into k parts) is p_k where $p_0 = 0.4, p_1 = 0.3, p_2 = 0.3$, find the probability that there is no particle in second generation. Assume that the particle act independently and identically irrespective of the generation.

5. A fair die is tossed once. Let A be the event that face 1,3 or 5 comes up, B be the event that it is 2,4 or 6; and C be the event that it is 1 or 6. Show that A and C are independent. Find $P(A, B$ or C occurs).

6. An urn contains four tickets marked with numbers $112, 121, 211, 222$ and one ticket is drawn at random. Let A_i $(i = 1, 2, 3)$ be the event that i^{th} digit of the number of the ticket drawn is 1. Discuss the independence of the events A_1, A_2 and A_3.

7. There are two identical boxes containing respectively 4 white and 3 red ball; 3 white and 7 red ball. A box is chosen at random and a ball is drawn from it. Find the probability that the ball is white. If the ball is white, what is the probability that it is from the first box.

8. The probability that an airplane accident which is due to structure failure is diagnosed correctly is 0.85 and the probability that an airplane accident which is not due to structure failure is diagnosed correctly as due to structure failure is 0.15 . If 30% of all airplane accidents are due to structure failure, find the probability that an airplane accident is due to structure failure given that it has been diagnosed is being to structure failure.

9. The numbers $1, 2, 3, \ldots, n$ are arranged in random order. Find the probability that the digits $1, 2, \ldots, k$ $(k < n)$ appear as neighbors in that order.

10. In a town of $(n+1)$ inhabitants, a person tells a rumor to a second person, who in turn, repeats it to a third person etc. At each step, the recipient of the rumor is chosen at random from the n people available. Find the probability that the rumor will be told r times without returning to the originator.

11. A secretary has to send n letters. She write addresses on n envelopes and absentmindedly places letters one in each envelope. Find the probability that at least one letter reaches the correct destination.

12. A pond contains red and golden fish. There are 3000 red and 7000 golden fish, of which 200 and 500 respectively, are tagged. Find the probability that a random sample of 100 red and 200 golden fish will show 15 and 20 tagged fish, respectively.

13. A coin is tossed 4 times. Let X denote the number of times a head is followed immediately by a tail. Find the distribution, mean and variance of X.

14. Let X be a random variable with mean μ and variance σ^2. Show that $E[(aX-b)^2]$, as a function of b, is minimized when $b=\mu$.

15. In a bombing attack there are 50 percent chance that a bomb can strike the target. Two hits are required to destroy the target completely. How many bombs must be dropped to give a 99 percent chance or better of completely destroying the target.

16. Let the probability density function of X be given by

$$f(x)=\begin{cases} c(4x-2x^2), & 0<x<2 \\ 0, & \text{otherwise} \end{cases}$$

   (a) What is the value of c?
   (b) What is the distribution of X?
   (c) $P\left[\frac{1}{2}<X<\frac{3}{2}\right]$?

17. A bombing plane flies directly above a railroad track. Assume that if a larger(small) bomb falls within 40(15) feet of the track, the track will be sufficiently damaged so that traffic will be disrupted. Let X denote the perpendicular distance from the track that a bomb falls. Assume that

$$f_X(x)=\frac{100-x}{5000}I_{[0,100]}(x).$$

   (a) Find the probability that a larger bomb will disrupt traffic.
   (b) If the plane can carry three large(eight small) bombs and uses all three(eight), what is the probability that traffic will be disrupted?

18. A random variable has the following probability distribution function

x	0	1	2	3	4	5	6	7	8
$p(x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	$15k$	$17k$

   (a) Determine the value of k.
   (b) Find $P(X<4)$, $P(X\geq 5)$, and $P(0<X<4)$.
   (c) Find the distribution function of X.
   (d) Find the smallest value of x for which $P(X\leq x)=1/2$.

19. An urn contains n cards numbered $1,2,\ldots,n$. Let X be the least number on the card obtained when m cards are drawn without replacement from the urn. Find the probability distribution of random variable X. Compute $P(X\geq 3/2)$.

20. Let $\Omega=[0,1]$. Define $X:\Omega\to\mathbb{R}$ by

$$X(w)=\begin{cases} w, & 0\leq w\leq 1/2 \\ w-1/2, & 1/2\leq w\leq 1 \end{cases}$$

   For any interval $I\subseteq[0,1]$, let $P(I)=\int_I 2x\,dx$. Determine the distribution function of X and use this to find $P(X>1/2),\ P(1/4<X<1/2),\ P(X<1/2|X>1/4)$.

21. If X be binomially distributed with $n=25$ and $p=0.2$. Find expectation, variance and $P\left[X<E(X)-2\sqrt{Var(X)}\right]$.

22. Let X be a Poisson distributed random variable such that $P[X = 0] = 0.5$. Find the mean of X.

23. In a uniform distribution, the mean and variance are given by 0.5 and $\frac{25}{12}$ respectively. Find the interval on which the probability is uniformly distributed.

24. Let

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - 2e^{-x} + e^{-2x}, & x \geq 0 \end{cases}$$

Is F_X a distribution function?. What type of random variable is X? Find the pdf of X?

25. A random number chosen from the interval $[0, 1]$ by a random mechanism. What is the probability that (i) its first decimal will be 3 (ii) its second decimal will be 3 (iii) its first two decimal will be 3's?.

26. Suppose that diameters of a shaft s manufactured by a certain machine are normal random variables with mean 10 and s.d. 0.1. If for a given application the shaft must meet the requirement that its diameter falls between 9.9 and 10.2 centimeters. What proportion of shafts made by this machine will meet the requirement.

27. A machine automatically packs a chemical fertilizer in polythene packets. It is observed that 10% of the packets weigh less than 2.42 kg while 15% of the packets weigh more than 2.50 kg. Assuming that the weight of the packet is normally distributed, find the mean and variance of the packet.

28. Let (X, Y) have a joint pdf $f(x, y) = 2, 0 < x < y < 1$. Find the joint distribution of Z = X/Y, and Y. Show that they are independent. Find their individual pdfs.

29. Let X and Y be independent random variables distributed as $U(0, 1)$. Find the distributions of $X + Y$, $X - Y$, $\mid X - Y \mid$, XY, X/Y, min (X, Y) and max (X, Y).

30. With respect to the above problem find the joint pdf of $(X + Y, X - Y)$. Hence or otherwise find the pdf of $(X + Y)/(X - Y)$.

31. Let X, Y be independent Gamma distributed random variable such that $X \sim \Gamma(\lambda, \alpha)$ and $Y \sim \Gamma(\lambda, \beta)$. Show that $U = X + Y$ and $V = X/(X + Y)$ are independently distributed. Find also their individual pdfs. Find the distribution of X/Y. Identify the distribution.

32. Let $X \sim N(0, 1)$ and $Y = X^2$. Find the pdf of Y. Identify this distribution. Let $Z = X_1^2 + X_2^2 + \ldots + X_n^2$, where X_i's are iid $N(0, 1)$. Using the above distribution, find the pdf of Z. Also identify the distribution.

33. Let X and Y be independent $N(0, 1)$. Find the distribution of $\frac{X}{Y}$, $\frac{X}{|Y|}$ and $\frac{|X|}{|Y|}$. Do the above if $X \sim N(0, \sigma_1^2)$ and $Y \sim N(0, \sigma_2^2)$.

34. Let X, Y be iid with pdf $f(x) = \alpha e^{-\alpha(x-\beta)}, x \geq \beta, \alpha > 0$. Find the distribution of $X - Y$.

35. A variable X is said to have log-normal distribution with parameters (μ, σ^2) if $\log X \sim N(\mu, \sigma^2)$. Find the pdf of X. Calculate its mean and variance.

36. If $X \sim \Gamma(\lambda, \alpha)$ and $Y \sim \Gamma(\lambda, \beta)$ find the distribution of $\frac{X}{Y}$ and $\frac{X/\alpha}{Y/\beta}$

37. If $X \sim N(0, 1)$ and $Y \sim \chi^2(n)$, find the distribution of $\frac{X}{\sqrt{(Y/n)}}$.

38. If $X_1, X_2, \ldots, X_n$ are iid $U(0, 1)$ and $P = X_1 X_2 \ldots X_n$, then show that $-2 \ln P \sim \chi^2(2n)$.

39. If $X_i \sim N(\mu_i, \sigma_i^2)$ $i = 1, 2$ are independent random variables, find the distribution of $Z^2 = \frac{(X_1 - \mu_1)^2}{(X_2 - \mu_2)^2}$

40. Suppose that the number of customers who visit SBI, IIT Delhi on a Saturday is a random variable with $\mu = 75$ and $\sigma = 5$. Find the lower bound for the probability that there will be more than 50 but fewer than 100 customers in the bank?

41. Does the random variable X exist for which

$$P\left[\mu - 2\sigma \leq X \leq \mu + 2\sigma\right] = 0.6 \ .$$

42. Suppose that the life length of an item is exponentially distributed with parameter 0.5. Assume that 10 such items are installed successively so that the i^{th} item is installed immediately after The $(I-1)$th item has failed. Let T_i be the time to failure of the i^{th} item $i = 1, 2, \ldots, 10$ and is always measured from the time of installation. Let S denote the total time of functioning of the 10 items. Assuming that T_i's are independent, evaluate $P(S \geq 15.5)$.

43. A certain industrial process yields a large number of steel cylinder Whose lengths are distributed normally with mean 3.25 inches and Standard deviation 0.05 inches. If two such cylinders are chosen at Random and placed end to end what is the probability that their Combined length is less than 6.60 inches?

44. A complex system is made of 100 components functioning Independently. The probability that any one component will fail During the period of operation equal 0.10. In order for the entire System to function at least 85 of the components must be working. Compute approximate probability of this.

45. Suppose that $X_I, I = 1, 2, \ldots, 450$ are independent random Variables, each having a distribution $N(0,1)$. Evaluate $P(X_1^2 + X_2^2 + \ldots + X_{450}^2 > 495)$ approximately.
($\Phi(2) = 0.9772$, $\Phi(1.5) = 0.9452$)

46. A computer is adding number, rounds each number off to the nearest Integer. Suppose that all rounding errors are independent and Uniformly Distributed over (-0.5,0.5).
(a) If 1500 numbers are added, what is the probability that the Magnitude Of the total error exceeds 15?
(b) How many numbers may be added together in order that the Magnitude of the total error is less than 10 with probability 0.90?

47. Let $X \sim b(n,p)$. Use CLT to find n such that $P[X > n/2] \geq 1 - \alpha$. Calculate the value of n, when $\alpha = 0.90$ and $p = 0.45$. (Use $P[Z \leq 1.28] = 0.90$)

48. Items are produced in such a manner that the probability of item being defective is p (assume unknown). A large number of items say n are classified as defective or non-defective. How large should n be so that we may be 99% sure that the relative frequency of defective differs from p by less than 0.05?

49. A person puts some rupee coins into a piggy-bank each day. The number of coins added on any given day is equally likely to be 1, 2, 3, 4, 5 or 6, and is independent from day to day. Find an approximate probability that it takes at least 80 days to collect 300 rupees? Final answer can be in terms of $\Phi(z)$ where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$.