MAL 250(Probability and Stochastic Processes)
Tutorial Sheet No. 7

1. Determine the equilibrium probability distribution of a three state Markov chain whose transition probability matrix is $\begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$.

2. For $j = 0, 1, \ldots$, let $P_{jj+2} = v_j$ and $P_{j0} = 1 - v_j$, define the transition probability matrix of Markov chain. Discuss the character of the states of this chain.

3. Let 0 be an absorbing state and for $j > 0$, $P_{jj} = p$, $P_{jj-1} = q$ where $p + q = 1$. Find $f_{j0}^{(n)}$, the probability that absorption takes place exactly at n^{th} step given initial state is j. Also find the expectation of this distribution.

4. Consider one dimensional random walk on states $\{0, 1, 2, \ldots, k-1\}$ with reflecting barriers at 0 and $k-1$. If the particle is at state j, it can move one unit up to state $j+1$ with probability p or one unit down to state $j-1$ with probability $1-p$, at any time while executing random walk except when $j = 0$ or $k-1$. If the particle reaches 0 it stays there with probability $1-p$ or moves to state 1 with probability p and, similarly, if it ever reaches $k-1$, it stays there with probability p or moves to $k-2$ with probability q. Find the stationary distribution of the Markov chain representing the motion of the particle.

5. Suppose that a production process changes states, $\{0, 1, 2, 3\}$ in accordance with a Markov chain having transition probability matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix}$$

where states 0, 1 are acceptable states while 2, 3 are unacceptable states. If the production process is said to be "up" when it is in any acceptable state and "down" when it is in an unacceptable state. Determine the rate of breakdown (that is, the production process goes from up to down).

6. Show that if state i in a Markov chain is recurrent and state i does not communicate with state j, then $p_{ij} = 0$.

7. For the gambler rain model, let M_i denote the mean number of games that must be played unit gambler either goes broke or wins complete fortune N in this game, given that it starts with $i (i = 0, 1, 2, \ldots, N)$. Show that M_i satisfies

$$M_0 = M_N = 0, \quad M_i = 1 + pM_{i+1} + qM_{i-1}, \quad i = 1, 2, \ldots, N-1.$$

Hence show that

$$M_i = \begin{cases} i(N-i), & \text{if } p = 1/2 \\ \frac{i}{q-p} - \frac{N}{q-p}\frac{1-(q/p)^i}{1-\left(\frac{q}{p}\right)^N} & \text{if } p \neq \frac{1}{2} \end{cases}$$

8. At all times an urn contains N balls, some are white and remaining are black. At each stage a coin having probability p of landing heads is tossed. If head appears, then a ball is chosen at random and replaced by a white ball, while if tail appears then a ball is chosen at random and replaced by a black one. Let X_n, denote the number of white balls in the urn after the n^{th} stage. Show that $\{X_n, n = 1, 2, \ldots\}$ is Markov chain and classify its states as transient or recurrent. Also find the periods of these states.
   If $p = 1$, what is expected time until there are only white balls in the urn if initially there are i white balls in the urn?

9. Two communication satellites are placed in orbit. The lifetime of the the satellite is exponential with mean $\frac{1}{\mu}$. If one fails its replacement is sent up. The time necessary to prepare and send up a replacement is exponential with mean $\frac{1}{\lambda}$. Let X_t = the number of satellites in the orbit at time t. Assume $\{X_t, t \geq 0\}$ is a Markov process with state space $\{0, 1, 2\}$. Show that the rate matrix is given by

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{bmatrix}$$

   Write down the Kolmogorov forward and backward equations for the above process.

10. Suppose that a company has four operators serving a single telephone number. Anybody who calls while all four operators are busy will receive a busy signal. Let X_t = number of busy operators at time t. Assume arrival and individual service processes are Poisson with rates λ and μ. Determine rate matrix Λ and the forward Kolmogorov equations for the Markov Process.

11. The birth and death process is called a pure death process if λ_i=0 for all i. Suppose $\mu_i = i\mu$, $i = 1, 2, 3, \ldots$ and initially $X_0 = n$. Show that X_t has $B(n, p)$ distribution with $p = e^{-\mu t}$.

12. The birth and death process is called a birth process if μ_i=0 for every i. Suppose $\lambda_i = i\lambda$, $i = 1, 2, 3, \ldots$. If $X_0 = n$ show that

$$P(X_t = i/X_0 = n) = P_{ni}(t) = \binom{i-1}{n-1} e^{-n\lambda t}(1 - e^{-\lambda t})^{i-n}, \quad i \geq n.$$

13. A digital camera needs three batteries to run. You buy a pack of 6 batteries, install three of these batteries into the camera. Whenever a battery is drained, you immediately replace the drained battery with the one new battery from the available stock. Assume that each battery lasts for an amount of time that is exponentially distributed with mean $1/\mu$, independent of all other batteries. Eventually camera stops running, only two batteries will be left out in the camera that are not drained. Let $X(t)$ denote the number of batteries not drained at time t. Write down the Kolmogorov forward equations for the process $\{X(t), t \geq 0\}$. Find the expected time that your camera will be able to run with the pack of batteries bought.

14. Suppose that a man operates a small auto collision shop. The arrival process of cars needing repair is a Poisson process of rate λ. The man first bumps a car, then paints the car, and then starts on the next car. The length of time to bump a car is exponential with mean $\frac{1}{\mu_1}$ and length of time to paint a car is exponential with mean $\frac{1}{\mu_2}$. Model the shop as a Markov process $\{X_t, t \geq 0\}$ where $X_t = (i, j)$ if there are $i > 0$ cars in the shop and the car being repaired is in stage j. The stage $j = 0$ means the bumping stage and $j = 1$ means the printing stage. If the shop is empty, let $X_t = (0, 0)$. Determine rate matrix Λ.

15. Find the limiting probabilities of the Markov chain in Problem no. 9. For what proportion of time in a year will the communication system be out of action with no satellites in operation.

16. Find the limiting probabilities for $\{X_t, t \geq 0\}$ if its rate matrix is

$$Q = \begin{pmatrix} -10 & 6 & 4 \\ 1 & -3 & 2 \\ 8 & 1 & -9 \end{pmatrix}$$

17. Suppose we have n identical machines operating independently and serviced by a single repair crew. If a machine breaks down while another is being repaired it must wait its turn before repairs can start. Assume each machine has operating time exponential with mean $\frac{1}{\mu}$ and a repair time exponential with mean $\frac{1}{\lambda}$. Let $X_t =$ no. of machines in operating condition at time t. Model X_t as a Markov chain with state space $E = \{0, 1, 2, \ldots, N\}$. Determine rate matrix Λ and the forward Kolmogorov equations. Determine the limiting equilibrium probability distribution of the process.

18. A cable car starts off with n riders. The times between successive stops of the car are independent exponential random variables, each with rate λ. At each stop, one rider gets off. This takes no time and no additional riders get on. Let $X(t)$ denote the number of riders present in car at time t. Write down the Kolmogorov forward equations for the process $\{X(t), t \geq 0\}$. Find the mean and variance of the number of riders present in car at any time t.

19. The arrival of large jobs at a server forms a Poisson process with rate two per hour. The service times of such jobs are exponentially distributed with mean 20 min. Only four jobs can be accommodated in the system at a time. Assuming that the fraction of computing power utilized by smaller jobs is negligible, determine the probability that a large job will be turned away because of lack of storage space. Also find the mean number of large jobs in the system at steady state.

20. Show that mean time spent in an $M/M/1$ system having arrival rate λ and service rate 2μ than the mean time spent in an $M/M/2$ system with arrival rate λ and each service rate μ.

21. Suppose that in an $M/M/1$ queue, the customer arrival rate is 3 per minute. Find the service rate so that 95% of the time the queue will contain less than 10 customers in steady state.

22. Ms. H. R. Cutt runs a one-person, unisex hair salon. She does not make appointments, but runs the salon on a first-come, first-served basis. She finds that she is extremely busy on Saturday mornings, so she is considering hiring a part-time assistant and even possibly moving to a larger building. Having obtained a master's degree in operations research (OR) prior to embarking upon her career, she elects to analyze the situation carefully before making a decision. She thus keeps careful records for a succession of Saturday mornings and finds that customers seem to arrive according to a Poisson process with a mean arrival rate of 5/hr. Because of her excellent reputation, customers were always willing to wait. The data further showed that customer processing time (aggregated female and male) was exponentially distributed with an average of 10 min. Cutt interested to calculate the following measures:
(a) What is the average number of customers in the shop?
(b) What is the average number of customers waiting for a haircut?
(c) What is the percentage of time an arrival can walk right in without having to wait at all?
(d) If waiting room has only four seats at present, what is the probability that a customer, upon arrival, will not able to find a seat and have to stand.

23. City Hospital's eye clinic offers free vision tests every Wednesday evening. There are three ophthalmologists on duty. A test takes, on the average, 20 min, and the actual time is found to be approximately exponentially distributed around this average. Clients arrive according to a Poisson process with a mean of 6/hr, and patients are taken on a first-come, first-served basis. The hospital planners are interested in knowing:
(a) What is the average number of people waiting?
(b) What is the average amount of time a patient spends at the clinic?
(c) What is the average percentage idle time of each of the doctors?

24. Consider an automobile emission inspection station with three inspection stalls, each with room for only one car. It is reasonable to assume that cars wait in such a way that when a stall becomes vacant, the car at the head of the line pulls up to it. The station can accommodate at most four cars waiting (seven in the station) at one time. The arrival pattern is Poisson with a mean of one car every minute during the peak periods. The service time is exponential with mean 6 min. The chief inspector wishes to know the average number in the system during peak periods, the average wait (including service), and the expected number per hour that cannot enter the station because of full capacity.

25. A rural telephone switch has C circuits available to carry C calls. A new call is blocked if all circuits are busy. Suppose calls have duration which has exponential distribution with mean $1/\mu$ and interarrival time of calls is exponential with mean $1/\lambda$. Assume calls arrive independently and are served independently. Model this process as a birth-death process and write the forward Kolmogorov equation for the process. Also find the probability that a call is blocked when the system in steady state.