Optimal portfolio trading subject to stochastic dominance constraints under second-order autoregressive price dynamics

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Abstract
This paper studies the optimal portfolio trading problem under the generalized second-order autoregressive execution price model. The problem of minimizing expected execution cost under the proposed price model is formulated as a quadratic programming (QP) problem. For a risk-averse trader, problem formulation under the second-order stochastic dominance constraints results in a quadratically constrained QP problem. Under some conditions on the execution price model, it is proved that the portfolio trading problems for risk-neutral and risk-averse traders become convex programming problems, which have many theoretical and computational advantages over the general class of optimization problems. Extensive numerical illustrations are provided, which render the practical significance of the proposed execution price model and the portfolio trading problems.

Keywords: optimal trading; execution cost; stochastic dominance; autoregressive behavior; convex programming; risk aversion

1. Introduction and motivation
Institutional investors and active fund managers need to rebalance their portfolios from time to time to ensure good returns from the investment and to keep a check on the risk level. Rebalancing of a portfolio results in the portfolio trading problem that should be accomplished efficiently to avoid incurring high execution cost. Trading a large volume of shares of an asset in the portfolio at once has a significant influence on its as well as other asset’s execution price. Apart from an asset’s trade, some other factors such as changing market conditions, sudden availability of some information about the asset (e.g., collective market information factor) also contribute to its price impact. The immediate execution demand of a trade results in a temporary change in price, and thus is called the temporary price impact, whereas the permanent changes in price due to supply–demand imbalance and the market information factor is termed the permanent price impact. The prevailing price of an asset without any impact factor (trading order or market information factor) is called its
no-impact price. The execution price of an asset consists of no-impact price dynamics and price impact function. Considering execution price paths of all the assets in the portfolio together by involving their cross-stock trading impacts and market information factor gives the portfolio execution price path.

Many possible models for portfolio execution price path, depending on different specifications of no-impact price dynamics and price impact functions, have been proposed in the literature (Bertsimas et al., 1999; Almgren and Chriss, 2000; Huberman and Stanzl, 2005; Gatheral and Schied, 2011; Khemchandani et al., 2013; Moazeni et al., 2013; Cheridito and Sepin, 2014). Bertsimas et al. (1999) presented two price models for portfolio referred therein as “linear percentage temporary price impact model (LPT portfolio price model)” and “linear price impact with information price model.” The latter has no-impact price dynamics as arithmetic random walk, and the permanent price impact function as a linear function of trade size and market information factor. In the LPT portfolio price model, the no-impact price dynamics is assumed as geometric Brownian motion, whereas the price impact function, as a percentage of no-impact price, is assumed to be linear in trade size and market information factor. Almgren and Chriss (2000) and Cheridito and Sepin (2014) modeled no-impact price dynamics as an arithmetic random walk. They have incorporated both temporary and permanent impact functions that are considered as linear in the rate of trading (number of shares traded in a period). In Huberman and Stanzl (2005), the no-impact price in any period is modeled as a convex combination of no-impact price and the execution price of the last trading period, whereas the price impact function is assumed to be a linear function in trade size, and a random factor that accounts for the price impact of other traders’ random trade orders in the market. Gatheral and Schied (2011) studied the optimal trading problem in Almgren–Chriss framework under geometric Brownian motion assumption for the underlying no-impact price process. Khemchandani et al. (2013) proposed price dynamics comprising weighted temporary and permanent price impact functions, and no-impact price path as geometric Brownian motion. Moazeni et al. (2013) used a jump-diffusion model wherein the price impact from random arrivals of other traders’ buy and sell orders is modeled by the two independent compound Poisson processes.

It is evident that in the literature often no-impact price dynamics in discrete time setting is modeled as an arithmetic random walk. That is, the no-impact price of the portfolio in any time period is assumed to be dependent on its price in the just preceding time period (first-order autoregressive [AR(1)] behavior). In this paper, we make an attempt to extend the linear price impact with information price model of Bertsimas et al. (1999), referred henceforth as AR(1) portfolio price model, to a more general framework by incorporating the second-order autoregressive (AR(2)) behavior in its no-impact price dynamics. The proposed price dynamics is referred to as the AR(2) portfolio price model. The motivation behind such an extension is based on the studies that infer that the prices of financial assets in any period show dependence not only on the last period, but also on the periods further in the past (Marcek, 2000; Rudoy and Rohrs, 2008; Rudoy, 2009). Rudoy and Rohrs (2008) and Rudoy (2009) studied the portfolio optimization problem under cointegrated vector autoregressive price model for the portfolio. Marcek (2000) presented a comparison of the forecasts of prices obtained from the fuzzy neural network model and the autoregressive model. Moreover, in this paper empirical results of statistical tests conducted on the real-market financial data is given in order to depict the presence of higher order autoregressive behavior in many financial time series.
Apart from generalizing portfolio execution price model from AR(1) to AR(2), in this paper we further show that under the AR(2) portfolio price model, the expected execution cost is a convex function of trading strategy. In view of this, the corresponding optimal portfolio trading problems under the AR(2) price model become convex. As it is well known that the class of convex programming problems exhibits many nice properties, for example, any local minimum is a global minimum, in this paper the generalization of price path does not add any computational complexity in the optimal trading problem. Moreover, with computational experiments it is shown that on many occasions the optimal trading strategy under the AR(2) portfolio price model exhibits lower cost of execution as compared to the strategy under the AR(1) price model.

To reduce the effect of price impact, a trader splits the trade of each asset in the portfolio into smaller parts to be traded periodically before some fixed time points in the future known as their execution times. A particular division of the trades of each asset in the portfolio is known as a portfolio trading strategy. The optimal portfolio trading problem refers to the problem of finding a trading strategy that gives minimum cost of execution when trade is to buy all the assets in the portfolio. However, where trade is to sell all the assets of the portfolio, the optimal trading strategy means the strategy that generates the highest revenue. The literature that studies the optimal trading problem includes Bertsimas and Lo (1998), Bertsimas et al. (1999), Almgren and Chriss (2000), Huberman and Stanzl (2005), Moazeni et al. (2010), Almgren (2012), Khemchandani et al. (2013), Cheridito and Sepin (2014), and Fruth et al. (2014). Under different specifications of execution price path, Bertsimas et al. (1999) discussed the unconstrained optimal portfolio trading problem of minimizing expected execution cost. The central feature of the study of Almgren and Chriss (2000) was the construction of an efficient portfolio trading frontier in a two-dimensional space with axes as expectation and variance of execution cost. Moazeni et al. (2010) studied the dependence of optimal portfolio trading strategies on the errors of estimating impact matrices. Huberman and Stanzl (2005) studied the effect of volatility, liquidity, trade duration, and speed of price reversion on the trading behavior. They examined how price correlations and cross-price impacts in portfolio trading affect the optimal execution cost. The literature that addresses the single-asset optimal trading problem with nonconstant liquidity and volatility includes Almgren (2012), Fruth et al. (2014), and Cheridito and Sepin (2014).

In this paper, an optimal portfolio trading problem under the AR(2) portfolio price model for the case in which the portfolio to be acquired is a combination of buy and sell orders in the assets is considered. In view of the fact that different assets in the portfolio need not have the same time of execution (Ait-Sahalia et al., 2008), in the problem formulation different execution times of assets is allowed. The different liquidities of assets is another factor taken into account. The formulation of the optimal portfolio trading problem results in a quadratic programming (QP) problem, which is further proved as a convex programming problem under some conditions on the AR(2) portfolio price model. In the proposed QP problem, the no-in-between sell constraints (no short-selling constraints) on the assets in the portfolio, which have buy orders, are imposed. This is required as there is no meaning of selling some shares of an asset in any time period if the trader intends to buy some fixed block of shares of that asset by its time of execution (Bertsimas and Lo, 1998; Bertsimas et al., 1999). Similarly, the constraints of no-in-between buy of shares are imposed on assets that are desired to be sold. Bertsimas et al. (1999) presented the closed-form solution of the unconstrained optimal trading problem of minimizing expected execution cost of portfolio in which the trade is to buy all the assets. They used the dynamic programming (DP) method introduced by Bellman.
(1957) to solve the problem in which short selling was allowed. However, at the same time they pointed out some challenges in standard techniques to solve dynamic optimization in the presence of constraints, and suggested some approximation methods to be used. They briefly mentioned the “static approximation method” to solve the constrained optimal trading problem. In this paper, we attempt to apply the static approximation method to solve the constrained portfolio trading problem with no-in-between sell and no-in-between buy restrictions on the respective assets.

By splitting a trading order into smaller packages to be traded periodically, a trader is exposed to long trading duration. This results in timing risk, which is the risk that occurs due to unfavorable price movements of the assets in long trading duration (Bertsimas and Lo, 1998; Almgren and Chriss, 2000). Thus for a risk-averse trader, it is plausible to minimize a measure for the timing risk besides minimizing expected execution cost. With different risk criteria, many mean-risk optimal trading problems have been studied in the literature (Almgren and Chriss, 2000; Huberman and Stanzl, 2005; Gatheral and Schied, 2011; Almgren, 2012; Moazeni et al., 2010, 2016; Cheridito and Sepin, 2014). In some cases, the risk criteria are put as the constraints in problem formulation, with mean of the distribution as the objective. The literature with CVaR and VaR-constrained problems include Birgin et al. (2011), Krokhmal et al. (2002), and Alexander and Baptista (2004). Although mean-risk models are convenient from a theoretical and computational point of view, making a choice of an appropriate risk criterion is difficult in some cases. A particular choice of risk measure may ignore some important characteristics of execution cost distribution. Another direction of decision making under risk is based on the use of stochastic dominance, in particular second-order stochastic dominance (SSD), in problem formulation. The importance of stochastic dominance, especially SSD, is well recognized in the area of portfolio optimization (Dentcheva and Ruszczyński, 2003, 2006; Roman et al., 2006; Sharma and Mehra, 2016), whereas in the area of optimal trading, single asset, or portfolio, the literature involving concept of stochastic dominance is limited. Khemchandani et al. (2016) incorporated constraints based on SSD (SSD-constraint) in the single-asset optimal trading problem. However, the applicability of stochastic dominance for the portfolio trading problem has not yet been explored. With this motivation, in this paper we make an attempt to study the optimal portfolio trading problem under SSD criterion. For a risk-averse trader, the SSD-constraint ensures that a trading strategy with lesser right tail distribution of execution cost is always preferred over the strategy with higher right tail distribution. Under the AR(2) portfolio price model, the formulation of optimal portfolio trading problem subject to SSD-constraint results in a quadratically constrained QP (QCQP) problem. The QCQP problem reduces to a convex programming problem under some conditions on price model.

In summary, in this paper we propose the AR(2) portfolio price model that is an extension of the AR(1) portfolio price model of Bertsimas et al. (1999). Under the AR(2) portfolio price model, the optimal portfolio trading problem of minimizing expected execution cost (problem (P1)) is formulated as a QP problem. For a risk-averse investor, the trading problem subject to SSD-constraint (problem (P2)) is studied, which under the AR(2) portfolio price model is formulated as the QCQP problem. Further, the QP and QCQP problems are proved to belong the class of convex programming problems. With numerical illustration the efficacy of AR(2) portfolio price model, and QP and QCQP problems are given.

The remainder of the paper is structured as follows. In Section 2, the general optimal portfolio trading problem is framed. In Section 3, the AR(2) portfolio price model is introduced. Further in this section, the portfolio execution cost under the AR(2) portfolio price model is formulated...
as a quadratic function of trading strategy, and its properties are discussed. The optimal portfolio trading problem subject to SSD-constraint is presented in Section 4. Section 5 presents numerical illustrations, and Section 6 concludes the paper.

2. Optimal portfolio trading problem

Consider a trader who wants to trade some fixed amount of shares $S_1, S_2, \ldots, S_N$ of $N$ assets by the time periods $T_1, T_2, \ldots, T_N$, respectively. We assume that the trading takes place at the end of a time period and trading of $n$th asset should be finished by the $T_n$th time period. Further, let $T = \max\{T_1, T_2, \ldots, T_N\}$. That is, $T$ denotes the time of acquisition of complete portfolio by which the desired portfolio position $[S_1, S_2, \ldots, S_N]'$ (denote the column vector by $\mathbf{S}$) in the $N$ assets must be acquired. Furthermore, we consider $S_n; n = 1, 2, \ldots, N$ to be of any sign positive or negative with the convention that positive $S_n$ for some $n$ signifies that the trader wants to buy $S_n$ shares of $n$th asset within $T_n$ number of time periods, whereas negative $S_n$ for some $n$ indicates that the trader wants to sell $|S_n|$ shares of $n$th asset in the portfolio by $T_n$ number of time periods. Let $P_{nt}$ and $S_{nt}$ be the execution price and the number of shares of the $n$th asset traded at the end of $t$th time period, respectively. Thus, the column vectors $\mathbf{P}_t (= [P_{1t}, P_{2t}, \ldots, P_{Nt}]')$ and $\mathbf{S}_t (= [S_{1t}, S_{2t}, \ldots, S_{Nt}'])$ denote the execution price of portfolio and the portfolio position by the end of $t$th time period. Consider the following matrix of order $N \times T$, which gives division of each asset in their time period of execution:

$$
[\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_T] = 
\begin{pmatrix}
S_{11} & S_{12} & \cdots & S_{1T} \\
S_{21} & S_{22} & \cdots & S_{2T} \\
\vdots & \vdots & \ddots & \vdots \\
S_{N1} & S_{N2} & \cdots & S_{NT}
\end{pmatrix}_{N \times T},
$$

where $S_{nt} = 0 \quad \forall \quad t > T_n, \ n = 1, 2, \ldots, N$.

The execution price vector $\mathbf{P}_t$ is a random process that changes in each time period owing to price impact of trade size, realized market information factor in that period, and changes in other factors. A trader tends to minimize expected execution cost such that acquisition of desired portfolio is completed by time period $T$. Without loss of generality, we assume that the first $N'$ assets in the portfolio of $N$ assets are to be bought, whereas the last $N - N'$ assets are to be sold by their respective time of acquisition. That is,

$$
\mathbf{S}_n > 0 \quad \forall \quad n = 1, 2, \ldots, N' \quad \text{and} \quad \mathbf{S}_n < 0 \quad \forall \quad n = N' + 1, N' + 2, \ldots, N.
$$

Mathematically, we consider following portfolio trading problem.

\begin{equation}
\min_{\{\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_T\}} \mathbb{E}_1\left(\sum_{t=1}^{T} \mathbf{P}_t \mathbf{S}_t\right)
\end{equation}
subject to
\[
\sum_{t=1}^{T} S_{nt} = \xi_n \quad \forall n = 1, 2, \ldots, N, \tag{2}
\]
\[
S_{nt} = 0 \quad \forall t = T_n + 1, \ldots, T; \quad n = 1, 2, \ldots, N, \tag{3}
\]
\[
S_{nt} \geq 0 \quad \forall t = 1, 2, \ldots, T_n; \quad n = 1, 2, \ldots, N', \tag{4}
\]
\[
S_{nt} \leq 0 \quad \forall t = 1, 2, \ldots, T_n; \quad n = N' + 1, N' + 2, \ldots, N, \tag{5}
\]

where $E_t$ (in particular $E_1$ in problem (P1)) represents the conditional expectation given the realization of execution price dynamics $P_t$ up to the $t$th time period. The objective function, given by (1), is to minimize expected cost of execution of the portfolio $[E_1(\sum_{i=1}^{T} P_i S_i) = E_1(\sum_{i=1}^{T} \sum_{n=1}^{N} P_{ni} S_{ni})]$, which is an algebraic sum of expected cost of buying first $N'$ assets of the portfolio $[E_1(\sum_{i=1}^{T} \sum_{n=1}^{N'} P_{ni} S_{ni})]$ and the expected revenue generated by selling last $N - N'$ assets $[-E_1(\sum_{i=1}^{T} \sum_{n=N'+1}^{N} P_{ni} S_{ni})]$. Thus, if some trading strategy the expected execution cost of the portfolio is positive then cost of buying first $N'$ assets is more than the revenue generated by selling last $N - N'$ assets and vice versa.

The set of constraints (2) and (3) ensures that the trade of all the assets is completed by their respective time of execution. The set $\{t; \ \ t = T_n + 1, \ldots, T\}$ in constraint (3) is an empty set for assets that have their time of executions equal to $T$. The nonnegativity constraints (4) impose no-in-between sell restrictions on the first $N'$ assets, whereas nonpositivity constraints (5) impose no-in-between buy restrictions on the last $N - N'$ assets of the portfolio.

To complete the formulation of optimal portfolio trading problem (P1), one needs to specify the dynamics of execution price $P_t$. One of the specifications of $P_t$ is AR(1) portfolio price model, proposed by Bertsimas et al. (1999), which is given as follows:

\[
P_t = P_{t-1} + AS_t + BX_t + \epsilon_t, \tag{6}
\]

\[
X_t = CX_{t-1} + \eta_t, \tag{7}
\]

where $X_t$ is $M \times 1$ vector of the market information factor. The matrix $C$ is of order $M \times M$ and have eigenvalues less than unity. Symbol $\eta_t$ represents $M \times 1$ white noise vector with mean 0 (zero vector of order $M$) and $M \times M$ variance–covariance matrix $\sum_\eta$. Matrices $A$ and $B$ of orders $N \times N$ and $N \times M$, respectively, measure the impact of trade size and market information factor on the execution price. The $N \times 1$ order vector $\epsilon_t$ corresponds to white noise in price dynamics with mean 0 (zero vector of order $N$) and $N \times N$ order matrix $\sum_\epsilon$ as variance–covariance matrix. The matrices $A$, $B$, and $C$, and vectors $X_t$, $\epsilon_t$, and $\eta_t$ are explicitly given as follows:

\[
A = [a_{ij}]_{N \times N}, \quad B = [b_{ij}]_{N \times M}, \quad C = [c_{ij}]_{M \times M},
\]

\[
X_t = [X_1, X_2, \ldots, X_M]^t, \quad \epsilon_t = [\epsilon_1, \epsilon_2, \ldots, \epsilon_N]^t, \quad \eta_t = [\eta_1, \eta_2, \ldots, \eta_M]^t.
\]
Table 1
Result of estimation of fourth-order autoregressive equation fitted to the daily closing prices of Reliance Industries Ltd. stock \( P_t = u_1 P_{t-1} + u_2 P_{t-2} + u_3 P_{t-3} + u_4 P_{t-4} \) for each trading day of year 2014 using EViews statistical package

<table>
<thead>
<tr>
<th>Lag order ((i))</th>
<th>Coefficient ((u_i))</th>
<th>Standard error</th>
<th>t-Statistics</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.120894</td>
<td>0.065143</td>
<td>17.20676</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>-0.255558</td>
<td>0.097755</td>
<td>-2.614266</td>
<td>0.0095</td>
</tr>
<tr>
<td>3</td>
<td>0.160971</td>
<td>0.098036</td>
<td>1.641954</td>
<td>0.1019</td>
</tr>
<tr>
<td>4</td>
<td>-0.026269</td>
<td>0.065217</td>
<td>-0.402787</td>
<td>0.6875</td>
</tr>
</tbody>
</table>

Bertsimas et al. (1999) presented the closed-form solution of the optimal portfolio trading problem of minimizing expected execution cost under the price dynamics given by (6) and (7) without nonnegativity constraints. They obtained the solution by DP method for the case in which \( T_1 = T_2 = \ldots = T_N = T \) and \( S_1 = S_2 = \ldots = S_N \) with \( S_n > 0 \) for all \( n = 1, 2, \ldots, N \) (that is all assets are to be bought).

In the next section the AR(2) portfolio price model, formulation of execution cost under this model, and discussion of its properties are presented.

3. The AR(2) portfolio price model and cost of execution

The price dynamics of an asset, say \( n \)th asset, in the AR(1) portfolio price model, (6) and (7), is given as follows:

\[
P_{nt} = P_{n(t-1)} + \sum_{i=1}^{N} a_{ni} S_{it} + \sum_{i=1}^{M} b_{ni} X_{it} + \epsilon_{nt}. \tag{8}
\]

According to Equation (8), the price of \( n \)th asset at the end of \( t \)th time period depends on its price realized at the end of preceding time period \( t-1 \) (AR(1) behavior). We propose the generalized price model by considering AR(2) behavior where prices realized at the end of last two \((t-1)\)th and \((t-2)\)th time periods are considered. This extension enables to study the optimal portfolio trading problem with wider state space of execution price of portfolio. The motivation for such an extension depends on many theoretical studies of higher order autoregressive behavior in many financial time series (Bertsimas et al., 1999; Marcek, 2000; Rudoy and Rohrs, 2008; Rudoy, 2009). Moreover, we present empirical results of statistical tests conducted on the real market financial data of daily closing prices of Reliance Industries Ltd. stock drawn from NSE India (National Stock Exchange of India) database. We fit the fourth-order autoregressive equation to the considered real market data using EViews (Econometric Views) statistical package. The output obtained is shown in Table 1. The last column of the table represents \( p \)-value of the test. The underlying theory suggests the rejection of null hypothesis of zero coefficient of a lag order if the \( p \)-value is smaller than the significance level. It is evident from Table 1 that in the financial data considered, the coefficient of the first and second lag orders are significant at 5% level of significance, whereas coefficients corresponding to higher lag orders, third and fourth, are not significant.
In view of the theoretical and empirical studies of higher order autoregressive behavior, we propose following execution price dynamics of \( n \)th asset:

\[
P_{nt} = u_n P_{n(t-1)} + v_n P_{n(t-2)} + \sum_{i=1}^{N} a_{ni} S_{nt} + \sum_{i=1}^{M} b_{ni} X_{it} + \epsilon_{nt},
\]

(9)

where \( u_n \) and \( v_n \) give the degree of dependence of \( P_{nt} \) on \( P_{n(t-1)} \) and \( P_{n(t-2)} \), respectively. Under the AR(2) price path of \( n \)th asset, given by (9), the dynamics of AR(2) portfolio price model is given as follows:

\[
P_t = UP_{t-1} + VP_{t-2} + AS_t + BX_t + \epsilon_t,
\]

(10)

\[
X_t = CX_{t-1} + \eta_t,
\]

(11)

where \( U = \text{diag}([u_1, u_2, \ldots, u_N]) \) and \( V = \text{diag}([v_1, v_2, \ldots, v_N]) \) are the diagonal matrices.

In further discussion, we evaluate the execution cost of the portfolio under AR(2) portfolio price model, (10) and (11). Consider following notations:

\[
L_{2t} = \sum_{k=0}^{t} \binom{2t-k}{k} U^{2t-2k} v^k \forall t = 0, 1, \ldots, \left\lfloor \frac{T-1}{2} \right\rfloor,
\]

(12)

\[
L_{2t-1} = \sum_{k=0}^{t-1} \binom{2t-1-k}{k} U^{2t-1-2k} v^k \forall t = 1, 2, \ldots, \left\lfloor \frac{T}{2} \right\rfloor,
\]

(13)

where \( \lfloor x \rfloor \) gives the smallest integer greater than or equals to \( x \).

Remark 1. The matrices \( L_t, \ t = 0, 1, \ldots, T \) are diagonal matrices of order \( N \). We take \( L_t = \text{diag}([L_{1t}, L_{2t}, \ldots, L_{Nt}]) \), where

\[
L_{n(2t)} = \sum_{k=0}^{t} \binom{2t-k}{k} u_n^{2t-2k} v_n^k \forall t = 0, 1, \ldots, \left\lfloor \frac{T-1}{2} \right\rfloor,
\]

(14)

\[
L_{n(2t-1)} = \sum_{k=0}^{t-1} \binom{2t-1-k}{k} u_n^{2t-1-2k} v_n^k \forall t = 1, 2, \ldots, \left\lfloor \frac{T}{2} \right\rfloor,
\]

(15)

\( \forall n = 1, 2, \ldots, N \).

Lemma 1. By Equations (12) and (13), we get

\[
L_0 = I_T, \quad UL_0 = L_1,
\]

(16)

\[
UL_t + VL_{t-1} = L_{t+1} \forall t = 1, 2, \ldots, T,
\]

(17)

where \( I_T \) is an identity matrix of order \( T \).
In particular, \( \forall n = 1, 2, \ldots, N \) we have

\[
L_{n0} = 1, \quad u_n L_{n0} = L_{n1}, \quad (18)
\]

\[
u_n L_{nt} + v_n L_{n(t-1)} = L_{n(t+1)} \quad \forall t = 1, 2, \ldots, T. \quad (19)
\]

**Proof.** See Appendix A

**Lemma 2.** Under the dynamics of market information factor \( X_t \), (11),

\[
X_t = C^{t-1} X_1 + \eta_t + C \eta_{t-1} + \cdots + C^{t-2} \eta_2, \quad \forall t = 2, 3, \ldots, T. \quad (20)
\]

**Proof.** See Appendix B

**Lemma 3.** Under the AR(2) portfolio price model, (10) and (11),

\[
P_t = L_t P_0 + V L_{t-1} P_{t-1} + \left( \sum_{k=0}^{t-1} L_k B C^{t-1-k} \right) X_1 + \sum_{k=0}^{t-1} L_k A S_{t-k} + \sum_{k=0}^{t-1} L_k \epsilon_{t-k}
\]

\[
+ \sum_{k=0}^{t-2} \left( \sum_{r=0}^{k} L_r B C^{k-r} \right) \eta_{t-k} \quad \forall t = 2, 3, \ldots, T. \quad (21)
\]

**Proof.** See Appendix C

Next, it is shown that the execution cost of portfolio under AR(2) portfolio price model can be formulated as a quadratic function of trading strategy.

**Proposition 1.** Under the AR(2) portfolio price model, (10) and (11), the portfolio execution cost is given by

\[
\sum_{t=1}^{T} P_t S_t = W' S + S' Q S + E' S + F' S, \quad (22)
\]
where $S$, $W$, $E$, and $F$ are matrices of order $NT \times 1$ each, and $Q$ is a matrix of order $NT \times NT$, which are given as follows:

$$S = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_T \end{pmatrix}, \quad W = \begin{pmatrix} L_1P_0 + VL_0P_{-1} + \left( \sum_{k=0}^{0} L_kBC^{0-k} \right)X_1 \\ L_2P_0 + VL_1P_{-1} + \left( \sum_{k=0}^{1} L_kBC^{1-k} \right)X_1 \\ L_3P_0 + VL_2P_{-1} + \left( \sum_{k=0}^{2} L_kBC^{2-k} \right)X_1 \\ \vdots \\ L_TP_0 + VL_TP_{-1} + \left( \sum_{k=0}^{T-1} L_kBC^{T-1-k} \right)X_1 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 \\ \sum_{k=0}^{0} L_k \epsilon_{1-k} \\ \sum_{k=0}^{1} L_k \epsilon_{2-k} \\ \sum_{k=0}^{2} L_k \epsilon_{3-k} \\ \vdots \\ \sum_{k=0}^{T-1} L_k \epsilon_{T-k} \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ \sum_{k=0}^{0} \left( \sum_{r=0}^{k} L_rBC^{k-r} \right) \eta_{2-k} \\ \sum_{k=0}^{1} \left( \sum_{r=0}^{k} L_rBC^{k-r} \right) \eta_{3-k} \\ \vdots \\ \sum_{k=0}^{T-2} \left( \sum_{r=0}^{k} L_rBC^{k-r} \right) \eta_{T-k} \end{pmatrix}$$

and

$$Q = \begin{pmatrix} L_0A & \frac{1}{2}(L_1A)' & \frac{1}{2}(L_2A)' & \ldots & \frac{1}{2}(L_{T-2}A)' & \frac{1}{2}(L_{T-1}A)' \\ \frac{1}{2}L_1A & L_0A & \frac{1}{2}(L_1A)' & \ldots & \frac{1}{2}(L_{T-3}A)' & \frac{1}{2}(L_{T-2}A)' \\ \frac{1}{2}L_2A & \frac{1}{2}L_1A & L_0A & \ldots & \frac{1}{2}(L_{T-4}A)' & \frac{1}{2}(L_{T-3}A)' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2}L_{T-1}A & \frac{1}{2}L_{T-2}A & \frac{1}{2}L_{T-3}A & \ldots & L_1A & L_0A \end{pmatrix}$$
where $P_0$ denotes the current portfolio price vector and $P_{-1}$ is the portfolio price at the end of the last trading period. Symbol 0 represents $N \times 1$ zero column vector.

**Proof.** The execution cost function is

$$
\sum_{t=1}^{T} P'_i S_t = P'_1 S_1 + \sum_{t=2}^{T} P'_i S_t
$$

(by Equation (10) with $t = 1$ and by Equation (21) of Lemma 3)

$$
= \left[ U P_0 + VP_{-1} + AS_1 + BX_1 + \epsilon_t \right]' S_1 + \sum_{t=2}^{T} \left[ L_t P_0 + VL_{t-1} P_{-1} + \sum_{k=0}^{t-1} L_k BC^{t-1-k} \right] X_1 + \sum_{k=0}^{t-1} L_k AS_{t-k} + \sum_{k=0}^{t-1} L_k \epsilon_{t-k} \right]' S_t
$$

$$
= \sum_{t=1}^{T} \left[ L_t P_0 + VL_{t-1} P_{-1} + \left( \sum_{k=0}^{t-1} L_k BC^{t-1-k} \right) X_1 \right]' S_t + \sum_{t=1}^{T} \sum_{k=0}^{t-1} L_k AS_{t-k} \right]' S_t
$$

$$
+ \sum_{t=1}^{T} \sum_{k=0}^{t-1} \left[ L_k \epsilon_{t-k} \right]' S_t + \sum_{t=2}^{T} \sum_{k=0}^{t-2} \left[ \left( \sum_{r=0}^{k} L_r BC^{r-k} \right) \eta_{t-k} \right]' S_t
$$

$$
= W'S + S'QS + E'S + F'S.
$$

**Remark 2.** Owing to the presence of random vectors $\epsilon$ and $\eta$, the matrices $E$ and $F$ are random. We consider the expectation ($E_1$) of execution cost with respect to the first time period where initial price vectors $P_0$ and $P_{-1}$ and the market information factor $X_1$ are known.

$$
E_1 \left( \sum_{t=1}^{T} P'_i S_t \right) = E_1 \left( W'S + S'QS + E'S + F'S \right)
$$

$$
= W'S + S'QS + E_1 \left( E'S + F'S \right)
$$

$$
= W'S + S'QS + E_1 \left[ \sum_{t=1}^{T} \sum_{k=0}^{t-1} \left( L_k \epsilon_{t-k} \right)' S_t + \sum_{t=1}^{T} \sum_{k=0}^{t-2} \left[ \left( \sum_{r=0}^{k} L_r BC^{r-k} \right) \eta_{t-k} \right]' S_t \right]
$$

$$
= W'S + S'QS + \sum_{t=1}^{T} \sum_{k=0}^{t-1} E_1 (\epsilon_{t-k})' (L_k)' S_t + \sum_{t=1}^{T} \sum_{k=0}^{t-2} E_1 (\eta_{t-k})' \left( \sum_{r=0}^{k} L_r BC^{r-k} \right)' S_t
$$

$$
= W'S + S'QS.
$$

(23)
By (23), the expected execution cost is a quadratic function in decision variable vector $\mathbf{S}$. In view of this, (P1) becomes a QP problem as follows:

$$\text{(QP problem)} \quad \min_{\mathbf{S}} \quad \mathbf{W}'\mathbf{S} + \mathbf{S}'\mathbf{Q}\mathbf{S}$$

subject to

$$\sum_{t=1}^{T_n} S_{nt} = \bar{S}_n \quad \forall \ n = 1, 2, \ldots, N,$$

$$S_{nt} = 0 \quad \forall \ t = T_n + 1, \ldots, T; \ n = 1, 2, \ldots, N,$$

$$S_{nt} \geq 0 \quad \forall \ t = 1, 2, \ldots, T_n; \ n = 1, 2, \ldots, N',$$

$$S_{nt} \leq 0 \quad \forall \ t = 1, 2, \ldots, T_n; \ n = N' + 1, N' + 2, \ldots, N.$$

**Theorem 1.** If $\mathbf{A}$ is a diagonal matrix with positive diagonal elements, then the matrix $\mathbf{Q}$ is a positive definite matrix.

**Proof:** Before presenting the proof of Theorem 1, let us consider following lemma that makes the basis for the proof of the theorem.

**Lemma 4.** For all $n = 1, 2, \ldots, N$, the matrices

$$\mathbf{Q}_n = \begin{pmatrix}
L_{n0} & \frac{1}{2}L_{n1} & \frac{1}{2}L_{n2} & \cdots & \frac{1}{2}L_{n(T-2)} & \frac{1}{2}L_{n(T-1)} \\
\frac{1}{2}L_{n1} & L_{n0} & \frac{1}{2}L_{n1} & \cdots & \frac{1}{2}L_{n(T-3)} & \frac{1}{2}L_{n(T-2)} \\
\frac{1}{2}L_{n2} & \frac{1}{2}L_{n1} & L_{n0} & \cdots & \frac{1}{2}L_{n(T-4)} & \frac{1}{2}L_{n(T-3)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{2}L_{n(T-1)} & \frac{1}{2}L_{n(T-2)} & \frac{1}{2}L_{n(T-3)} & \cdots & \frac{1}{2}L_{n1} & L_{n0}
\end{pmatrix}$$

(24)

are positive definite matrices.

**Proof:** See Appendix D

**Proof of Theorem 1.** Consider matrix $\mathbf{A}$ as a diagonal matrix with positive diagonal elements. That is,

$$a_{ij} = 0 \quad \text{if} \ i \neq j \quad \text{and} \quad a_{ii} > 0 \quad \forall \ i, j = 1, 2, \ldots, T.$$

Then,

$$\mathbf{S}'\mathbf{Q}\mathbf{S} = a_{11}\tilde{\mathbf{S}}_1'\mathbf{Q}_1\tilde{\mathbf{S}}_1 + a_{22}\tilde{\mathbf{S}}_2'\mathbf{Q}_2\tilde{\mathbf{S}}_2 + \cdots + a_{NN}\tilde{\mathbf{S}}_N'\mathbf{Q}_N\tilde{\mathbf{S}}_N,$$

(25)

where $\tilde{\mathbf{S}}_n = [S_{n1}, S_{n2}, \ldots, S_{nT}]' \quad \forall \ n = 1, 2, \ldots, N.$

Since $\mathbf{Q}_n$ is a positive definite matrix and $a_{nn} > 0$, we have

$$\mathbf{S}_n' (a_{nn} \mathbf{Q}_n) \mathbf{S}_n > 0 \quad \forall \ \mathbf{S}_n \neq \mathbf{0}.$$

(26)
By (25) and (26),

\[ S'QS > 0 \text{ whenever } S_n \neq 0 \text{ for at least one } n \in \{1, 2, \ldots, N\}. \]

Thus, \( S'QS > 0 \forall S \neq 0 \). That is, \( Q \) is a positive definite matrix.

\[ \blacksquare \]

Remark 3. By Theorem 1, the QP problem is a convex programming problem for the special case when \( A \) is a diagonal matrix with positive diagonal elements.

Remark 4. The conditions on \( A \), given in Theorem 1, are sufficient but not necessary to make matrix \( Q \) a positive definite matrix.

Remark 5. This is the general behavior of financial market that the impact of buying some shares of a stock results in an increase in its price, whereas selling decreases its price. The condition of positivity of diagonal entries in \( A \) captures this effect.

Remark 6. The assumption of zero values of the off-diagonal entries of matrix \( A \) imposes the condition that the trading in one stock does not affect the prices of other stocks in the portfolio. An important point to note here is that although under this assumption on \( A \) the stock prices are independent from the impact of trading of one another, but at the same time this assumption does not imply that stocks are not correlated at all. The price dynamics of stocks in the portfolio are still correlated by the joint normally distributed noise factor \( (\epsilon_i) \) and the common market information factor \( (X_t) \).

Remark 7. The assumption of impact matrix \( A \) as the diagonal matrix can be permissible if the assets of the portfolio are not highly correlated. For example, we can expect less level of correlation between the assets from different financial sectors.

Remark 8. By Lemma 4, the single-asset optimal trading problem of minimizing expected execution cost under the AR(2) price model is a convex programming problem.

4. Optimal portfolio trading problem subject to SSD-constraint

In this section, first the basic definition of SSD in the context of any general distribution is given. This will be followed by stating some relations that are equivalent to SSD. Thereafter, the optimal portfolio trading problem subject to SSD-constraint is presented.

Consider \( X \) and \( Y \) as two random variables with distribution functions \( F_X \) and \( F_Y \), respectively. The variable \( X \) is said to dominate \( Y \) in SSD sense, which is denoted by \( X \geq_{SSD} Y \) (Fishburn, 1964; Sriboonchitta et al., 2009) iff

\[
\int_{-\infty}^{a} F_X(t)dt \leq \int_{-\infty}^{a} F_Y(t)dt \quad \forall a \in \mathbb{R}.
\]

The following three statements are equivalent in context of SSD (Whitmore and Findlay, 1978):

1. \( \int_{-\infty}^{a} F_X(t)dt \leq \int_{-\infty}^{a} F_Y(t)dt \quad \forall a \in \mathbb{R} \).
2. \( E([a - X]^+) \leq E([a - Y]^+) \quad \forall a \in \mathbb{R}, \) where \( x^+ = \max\{x, 0\} \).
3. $E[g(X)] \geq E[g(Y)]$ for any nondecreasing concave function $g(\cdot)$.

Let $X$ and $Y$ be discrete random variables each having $K$ equally probable outcomes. Let $(\alpha_1, \alpha_2, \ldots, \alpha_K)$ and $(\beta_1, \beta_2, \ldots, \beta_K)$ be the vectors of values taken by $X$ and $Y$, respectively, each with probability $\frac{1}{K}$. Without loss of generality, we can assume

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_K \quad \text{and} \quad \beta_1 \leq \beta_2 \leq \cdots \leq \beta_K. \quad (27)$$

By Roman et al. (2006), we have

$$X \succeq_{SSD} Y \quad \text{iff} \quad \frac{1}{i} \sum_{j=1}^{i} \alpha_j \geq \frac{1}{i} \sum_{j=1}^{i} \beta_j, \quad \forall \ i = 1, 2, \ldots, K, \quad (28)$$

with at least one strict inequality.

Since optimal portfolio trading problem (QP problem) is formulated with objective as to minimize the expectation of net cost of execution of portfolio, a trader would always prefer the strategy that gives less cost of execution compared to that of some reference strategy. Due to this formulation, the trading strategy ($S$) with negation of cost of execution dominating the negation of cost of execution of some reference strategy, in SSD sense, would be preferred. We consider reference strategy as the naive strategy ($N$), which is the equal division of trade of each asset in their time period of execution. That is, naive strategy $N$ is $NT \times 1$ column vector given as follows:

$$N = \begin{pmatrix}
N_1 \\
N_2 \\
N_3 \\
\vdots \\
N_T
\end{pmatrix},$$

where $N_t$ be the $N \times 1$ column vector $[N_{1t}, N_{2t}, \ldots, N_{Nt}]'$ with

$$N_{nt} = \frac{\bar{N}_n}{T_n} \quad \forall \ t = 1, 2, \ldots, T_n \quad \text{and} \quad N_{nt} = 0 \quad \forall \ t = T_n + 1, T_n + 2, \ldots, T, \ n = 1, 2, \ldots, N. \quad (29)$$

Let $\text{Cost}_S$ and $\text{Cost}_N$ be the costs of execution corresponding to the trading strategies $S$ and $N$, respectively. Then we require SSD-constraint as follows:

$$-\text{Cost}_S \succeq_{SSD} -\text{Cost}_N. \quad (30)$$

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The optimal portfolio trading problem with SSD-constraint (30) is as follows:

\[(P2) \min_S W'S + S'Q_S\]

subject to

\[\sum_{t=1}^{T_n} S_{nt} = S_n \forall n = 1, 2, \ldots, N,\]

\[S_{nt} = 0 \forall t = T_n + 1, \ldots, T; n = 1, 2, \ldots, N,\]

\[S_{nt} \geq 0 \forall t = 1, 2, \ldots, T_n; n = 1, 2, \ldots, N',\]

\[S_{nt} \leq 0 \forall t = 1, 2, \ldots, T_n; n = N' + 1, N' + 2, \ldots, N,\]

\[-\text{Cost}_S \geq \text{SSD} - \text{Cost}_N.\]

Next, we simplify constraint \(-\text{Cost}_S \geq \text{SSD} - \text{Cost}_N\) of problem (P2).

Consider \(K\) number of simulations of random vectors \([\epsilon_1, \epsilon_2, \ldots, \epsilon_T]\) and \([\eta_1, \eta_2, \ldots, \eta_T]\) of AR(2) portfolio price model, (10) and (11). Let \(E^i\) and \(F^i\) be the \(i\)th simulation of vectors \(E\) and \(F\), which correspond to \(i\)th simulation of \([\epsilon_1, \epsilon_2, \ldots, \epsilon_T]\) and \([\eta_1, \eta_2, \ldots, \eta_T]\), respectively. The corresponding \(K\) realizations of \(\text{Cost}_S\) and \(\text{Cost}_N\) are as follows:

\[\text{Cost}_S^i = W'S + S'Q_S + (E^i)'S + (F^i)'S, \quad i = 1, 2, \ldots, K,\]

\[\text{Cost}_N^i = W'N + N'Q_N + (E^i)'N + (F^i)'N, \quad i = 1, 2, \ldots, K.\]

By the equivalent definition of SSD, we have

\[-\text{Cost}_S \geq \text{SSD} - \text{Cost}_N \iff E([a - (-\text{Cost}_S)]^+) \leq E([a - (-\text{Cost}_N)]^+) \quad \forall a \in \mathbb{R}.\]

Further by Proposition 3.2 of Dentcheva and Ruszczynski (2003), if \(\{\text{Cost}_S^j; j = 1, 2, \ldots, K\}\) are the realizations of \(\text{Cost}_N\), then \(E([a - (-\text{Cost}_S)^+] \leq E([a - (-\text{Cost}_N)^+] \quad \forall a \in \mathbb{R},\) can be approximated by the following relation:

\[E([(-\text{Cost}_N^j) - (-\text{Cost}_S^j)]^+) \leq E([(-\text{Cost}_N^j) - (-\text{Cost}_S^j)]^+) \quad \forall j = 1, 2, \ldots, K,\]

which can be rewritten as follows:

\[E([\text{Cost}_S^j - \text{Cost}_N^j]^+) \leq E([\text{Cost}_S^j - \text{Cost}_N^j]^+) \quad \forall j = 1, 2, \ldots, K.\]

Consider following approximates

\[E([\text{Cost}_S^j - \text{Cost}_N^j]^+) \approx \frac{1}{K} \sum_{i=1}^{K} [\text{Cost}_S^i - \text{Cost}_N^i]^+;\]

\[\text{Cost}_S \geq \text{SSD} - \text{Cost}_N.\]

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\[
E([\text{Cost}_N - \text{Cost}_N^j]^+) \approx \frac{1}{K} \sum_{i=1}^{K} [\text{Cost}_N^i - \text{Cost}_N^j]^+.
\] (36)

From (33) to (36), it follows that in view of realization of \( K \) generated scenarios of \([\epsilon_1, \epsilon_2, \ldots, \epsilon_T]\) and \([\eta_1, \eta_2, \ldots, \eta_T]\), the SSD-constraint can be approximated by

\[
\frac{1}{K} \sum_{i=1}^{K} [\text{Cost}_N^i - \text{Cost}_N^j]^+ \leq \frac{1}{K} \sum_{i=1}^{K} [\text{Cost}_N^i - \text{Cost}_N^j]^+ \quad \forall j = 1, 2, \ldots, K.
\] (37)

To further simplify the SSD-constraint, let

\[
d_{ij} = [\text{Cost}_N^i - \text{Cost}_N^j]^+ \quad \forall i, j = 1, 2, \ldots, K.
\] (38)

By (37) and (38), we get following approximation of SSD-constraint:

\[
\frac{1}{K} \sum_{i=1}^{K} d_{ij} \leq \frac{1}{K} \sum_{i=1}^{K} [\text{Cost}_N^i - \text{Cost}_N^j]^+ \quad \forall j = 1, 2, \ldots, K.
\] (39)

Further, by definition \( x^+ = \max\{x, 0\} \), it follows that

\[
d_{ij} \geq 0 \quad \text{and} \quad d_{ij} \geq \text{Cost}_N^i - \text{Cost}_N^j \quad \forall i, j = 1, 2, \ldots, K.
\] (40)

From Equation (31), we get

\[
\text{Cost}_N^i - \text{Cost}_N^j = W'S + S'QS + (E^i)'S + (F^i)'S - \text{Cost}_N^j \quad \forall i, j = 1, 2, \ldots, K.
\] (41)

In view of (39)–(41), (P2) can be rewritten as follows:

(QCQP problem)

\[
\begin{align*}
\min_S & \quad W'S + S'QS \\
\text{subject to} & \quad \\
\sum_{t=1}^{T_n} S_{nt} &= \bar{S}_n \quad \forall n = 1, 2, \ldots, N, \\
S_{nt} &= 0 \quad \forall t = T_n + 1, \ldots, T; \quad n = 1, 2, \ldots, N, \\
S_{nt} &\geq 0 \quad \forall t = 1, 2, \ldots, T_n; \quad n = 1, 2, \ldots, N', \\
S_{nt} &\leq 0 \quad \forall t = 1, 2, \ldots, T_n; \quad n = N' + 1, \ldots, N, \\
\frac{1}{K} \sum_{i=1}^{K} d_{ij} &\leq \frac{1}{K} \sum_{i=1}^{K} [\text{Cost}_N^i - \text{Cost}_N^j]^+ \quad \forall j = 1, 2, \ldots, K, \\
d_{ij} &\geq W'S + S'QS + (E^i)'S + (F^i)'S - \text{Cost}_N^j \quad \forall i, j = 1, 2, \ldots, K, \\
d_{ij} &\geq 0 \quad \forall i, j = 1, 2, \ldots, K.
\end{align*}
\]

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The above-mentioned problem is a QCQP problem.

**Remark 9.** By Theorem 1, under the assumption on matrix \( A \) to be a diagonal matrix with positive diagonal elements, the QCQP problem is a convex programming problem.

## 5. Numerical illustrations

This section is further divided into three subsections. In Section 5.1, the algorithm to solve QP and QCQP problems by static approximation method is presented. This is followed by the discussion of a numerical example to demonstrate the practical significance of the AR(2) portfolio price model, and QP and QCQP problems. To present a simple numerical example, we consider financial market data provided in Bertsimas et al. (1999). We transform the parameters estimated for the LPT portfolio price model in Bertsimas et al. (1999) to the parameters of the AR(2) portfolio price model by appropriate scaling. In Section 5.2, the transformation of parameters is presented, which is followed by the discussion of the numerical example in Section 5.3.

### 5.1. Static approximation method

The market information variable \( X_t \) and noise factor \( \epsilon_t \) in the execution price dynamics \( P_t \) (Equation (10)) are external state variables that change due to their own natural dynamics. Because of the presence of external variables in the QP and QCQP problems, their optimal trading strategies cannot be determined in advance of trading. One requires to revise QP and QCQP problems in each time period according to new realizations by external variables (Bertsimas et al., 1999). Bertsimas et al. (1999) mentioned the static approximation method that takes into account the new realization of external variables in each time period, and enables one to include desired constraints in the formulation of optimal trading problem. They presented empirical analysis with some numerical examples, which infers that static approximation method performs sufficiently well in many cases. Khemchandani et al. (2013) used this method for solving their formulation of optimal trading problem. They compared solutions of optimal trading problem obtained by the static approximation method to the solution obtained by the well-known DP method.

Following is the algorithm of static approximation method to solve QP and QCQP problems. (The steps are being provided in context of QP problem. Same steps will be followed for QCQP problem.)

**Step 1:** For \( t = 1 \).
- Solve initial QP problem (say QP(1)) with initial input parameters as \( P_0, P_{-1}, X_1 \), and \( \bar{S} \) as the initial vector to be executed at the beginning. Solve QP(1) to obtain the optimal trading strategy as \( [S^{(1)}_1, S^{(1)}_2, \ldots, S^{(1)}_T] \).

**Step 2:** For \( t = 2 : T \).
- Implement \( S^{(t-1)}_{t-1} \) in the \( t \)th time period. Reformulate QP problem corresponding to the \( t \)th time period (QP(\( t \))) with input parameters as \( P_{t-1}, P_{t-2}, X_t \), and \( \bar{S} - \sum_{i=t}^{T} S^{(t)}_i \) as the portfolio to be executed in the time periods \( t \) through \( T \).

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Table 2
Ticker symbols, company names, and prices of the five considered stocks in the portfolio (source: Bertsimas et al. 1999)

<table>
<thead>
<tr>
<th>S. no. (n)</th>
<th>Ticker symbol</th>
<th>Company name</th>
<th>Price ($) ($P_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>AHP</td>
<td>AMER HOME PRODS</td>
<td>64.0625</td>
</tr>
<tr>
<td>2</td>
<td>AN</td>
<td>AMOCO</td>
<td>70.5000</td>
</tr>
<tr>
<td>3</td>
<td>BLS</td>
<td>BELL SOUTH</td>
<td>37.2500</td>
</tr>
<tr>
<td>4</td>
<td>CHV</td>
<td>CHEVRON</td>
<td>62.6250</td>
</tr>
<tr>
<td>5</td>
<td>DD</td>
<td>DUPON</td>
<td>88.9375</td>
</tr>
</tbody>
</table>

With the above Steps 1 and 2, we solve a sequence of $T$ QP problems $QP(t)$, $t = 1, 2, \ldots, T$, to obtain the final optimal trading strategy as $[S_1^{(1)}, S_2^{(2)}, \ldots, S_T^{(T)}]$. Similarly, the optimal trading strategy for QCQP problem can be obtained. In the present paper, the MOSEK solver within MATLAB software is used to solve the sequence of optimization problems $QP(t)$ and QCQP($t$), $t = 1, 2, \ldots, T$.

5.2. Parameter estimation

In Bertsimas et al. (1999), the estimation of parameters of LPT portfolio price model from the considered real market data is provided. The data considered are mainly drawn from three different sources. The primary source is the proprietary record of trades executed over the NYSE DOT system by trading desk at Investment Technology group (ITG) on every trading day between January 2, 1996 and December 31, 1996. The second is the NYSE TAQ data that extract quotes prevailing at time of ITG trades. The last is the S&P 500 tick data provided by the Tick Data, Inc., to get intraday levels for the S&P 500 index during 1996. For the numerical illustration purpose we consider the portfolio of five stocks listed in Table 2, which are selected from the list of 25 stocks given in Bertsimas et al. (1999). For the considered stocks, the LPT portfolio price path is assumed in Bertsimas et al. (1999), and the estimation of parameters corresponding to this price path is presented. By appropriate scaling, we transform the parameters estimated for the LPT portfolio price model in Bertsimas et al. (1999) to the parameters corresponding to AR(2) portfolio price model.

The LPT portfolio price model of Bertsimas et al. (1999) is given as follows:

$$P_t = \tilde{p}_t + \delta_t,$$

$$\tilde{p}_t = \exp(Z_t)\tilde{p}_{t-1},$$

$$\delta_t = \tilde{P}_t(A\tilde{P}_tS_t + BX_t),$$

$$X_t = CX_{t-1} + \eta_t,$$
Table 3
Estimates of expected return, standard deviation, and correlation coefficient (in percent per year) corresponding to \( Z_t \) of \( \tilde{p}_t \) (source: Bertsimas et al. 1999).

<table>
<thead>
<tr>
<th></th>
<th>AHP</th>
<th>AN</th>
<th>BLS</th>
<th>CHV</th>
<th>DD</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\mu} )</td>
<td>0.200</td>
<td>-0.103</td>
<td>-0.387</td>
<td>-0.167</td>
<td>0.283</td>
</tr>
<tr>
<td>( \hat{\sigma} )</td>
<td>0.268</td>
<td>0.196</td>
<td>0.283</td>
<td>0.222</td>
<td>0.225</td>
</tr>
</tbody>
</table>

\[
\begin{array}{cccccc}
AHP & AN & BLS & CHV & DD \\
\hline
AHP & 1.000 & 0.196 & 0.284 & 0.226 & 0.345 \\
AN  & 0.196 & 1.000 & 0.173 & 0.408 & 0.283 \\
BLS & 0.284 & 0.173 & 1.000 & 0.259 & 0.328 \\
CHV & 0.226 & 0.408 & 0.259 & 1.000 & 0.314 \\
DD  & 0.345 & 0.283 & 0.328 & 0.314 & 1.000 \\
\end{array}
\]

where \( \tilde{p}_t = [\tilde{P}_{1t}, \tilde{P}_{2t}, \ldots, \tilde{P}_{Nt}]' \) is the no-impact price path whose dynamics is given by Equation (43). Further, \( \bar{\delta}_t = [\delta_{1t}, \delta_{2t}, \ldots, \delta_{Nt}]' \) represents the price impact in \( t \)th time period. Further, \( \tilde{Z}_t \) is a diagonal matrix whose diagonal is a joint normal random vector \( \tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_N \) with mean vector \( \mu_Z \) and variance–covariance matrix \( \Psi_Z \). The \( \exp(\cdot) \) operator denotes the matrix exponential, which in this case, reduces to the element-wise exponential of the diagonal entries in \( Z_t \). All other symbols have the same meaning as in the AR(2) portfolio price model. The estimates of expected return, standard deviation, and correlation coefficient of five considered assets corresponding to price path \( \tilde{p}_t \) are listed in Table 3.

In the LPT portfolio price model, the no-impact price path is linear in log prices of assets. That is for \( n \)th asset from Equation (43) we have

\[
\ln(\tilde{P}_{nt}) = \ln(\tilde{P}_{n(t-1)}) + Z_{nt},
\]

where \( Z_{nt} \) is normally distributed with mean \( \mu_{Z_n} \) and variance \( \sigma_{Z_n}^2 \).

Further, we have the following no-impact price dynamics of the \( n \)th asset (denote by \( \tilde{P}_{nt} \)) under AR(2) portfolio price model.

\[
\tilde{P}_{nt} = U \tilde{P}_{n(t-1)} + V \tilde{P}_{n(t-2)} + \epsilon_{nt},
\]

where \( \epsilon_{nt} \) is normally distributed with mean \( \mu_{\epsilon_n} \) and variance \( \sigma_{\epsilon_n}^2 \).

In view of Equations (46) and (47), \( \epsilon_{nt} \), and \( Z_{nt} \) are the noise factors in the no-impact price and logarithm of no-impact price of \( n \)th asset. Therefore, we can consider \( \epsilon_n = P_n \epsilon^{Z_n} \). Let \( \rho_{Z_nZ_m} \) represent the correlation coefficient of \( Z_n \) and \( Z_m \), and \( \sigma'_{\epsilon_n\epsilon_m} \) be the annual covariance of \( \epsilon_n \) and \( \epsilon_m \).

\[
\begin{align*}
\sigma'_{\epsilon_n\epsilon_m} &= \text{cov}(\epsilon_n, \epsilon_m) \\
&= \text{cov}(P_n \epsilon^{Z_n}, P_m \epsilon^{Z_m}) \\
&= P_n P_m [E(\epsilon^{Z_n} \epsilon^{Z_m}) - E(\epsilon^{Z_n}) E(\epsilon^{Z_m})] \\
&= P_n P_m [E(\epsilon^{Z_n+Z_m}) - E(\epsilon^{Z_n}) E(\epsilon^{Z_m})]
\end{align*}
\]
Table 4
Annual variance–covariance matrix \( \Psi'_{\epsilon} \) (\( \text{S}^2 \))

<table>
<thead>
<tr>
<th></th>
<th>AHP</th>
<th>AN</th>
<th>BLS</th>
<th>CHV</th>
<th>DD</th>
</tr>
</thead>
<tbody>
<tr>
<td>AHP</td>
<td>489.8664</td>
<td>54.4180</td>
<td>46.4975</td>
<td>59.6350</td>
<td>206.3974</td>
</tr>
<tr>
<td>AN</td>
<td>54.4180</td>
<td>164.6183</td>
<td>16.4596</td>
<td>63.0741</td>
<td>98.5644</td>
</tr>
<tr>
<td>BLS</td>
<td>46.4975</td>
<td>16.4596</td>
<td>57.8057</td>
<td>23.4609</td>
<td>67.269</td>
</tr>
<tr>
<td>CHV</td>
<td>59.6350</td>
<td>63.0741</td>
<td>23.4609</td>
<td>149.0379</td>
<td>103.9396</td>
</tr>
<tr>
<td>DD</td>
<td>206.3974</td>
<td>98.5644</td>
<td>67.269</td>
<td>103.9396</td>
<td>760.9764</td>
</tr>
</tbody>
</table>

Table 5
Variance–covariance matrix \( \Psi_{\epsilon} \) corresponding to a trading period (\( \text{S}^2 \))

<table>
<thead>
<tr>
<th></th>
<th>AHP</th>
<th>AN</th>
<th>BLS</th>
<th>CHV</th>
<th>DD</th>
</tr>
</thead>
<tbody>
<tr>
<td>AHP</td>
<td>0.2368</td>
<td>0.0263</td>
<td>0.0225</td>
<td>0.0288</td>
<td>0.0998</td>
</tr>
<tr>
<td>AN</td>
<td>0.0263</td>
<td>0.0796</td>
<td>0.0080</td>
<td>0.0305</td>
<td>0.0476</td>
</tr>
<tr>
<td>BLS</td>
<td>0.0225</td>
<td>0.0080</td>
<td>0.0279</td>
<td>0.0113</td>
<td>0.0325</td>
</tr>
<tr>
<td>CHV</td>
<td>0.0288</td>
<td>0.0305</td>
<td>0.0113</td>
<td>0.0720</td>
<td>0.0502</td>
</tr>
<tr>
<td>DD</td>
<td>0.0998</td>
<td>0.0476</td>
<td>0.0325</td>
<td>0.0502</td>
<td>0.3678</td>
</tr>
</tbody>
</table>

By (48), an estimate of annual variance–covariance matrix \( \Psi'_{\epsilon} \) is given in Table 4. In Bertsimas et al. (1999), total 2069 half-hour trading periods are considered in a year. The matrix \( \Psi_{\epsilon} \) as variance–covariance matrix corresponding to per half-hour trading period is presented in Table 5.

Since the dynamics of market information factor is same in LPT and AR(2) portfolio price dynamics, we use the same estimate of \( B, C, \) and \( \sigma_\eta \) as in Bertsimas et al. (1999) with \( M = 1 \). Thus, we have

\[
M = 1, \quad C = 0.0354, \quad \sigma_\eta = \sqrt{1 - C^2} = 0.9994, \quad B = \begin{pmatrix}
3.74 \times 10^{-5} \\
2.28 \times 10^{-5} \\
0.26 \times 10^{-5} \\
11.70 \times 10^{-5} \\
2.38 \times 10^{-5}
\end{pmatrix}.
\]

Furthermore, the matrix \( A \) that appears in Equation (44) of LPT portfolio price path (e.g., \( A_{\text{LPT}} \)) gives percentage impact on the dollar value of the trade, whereas \( A \) present in AR(2) portfolio price dynamics, Equation (10), gives the linear price impact. Thus, one needs to scale the estimated matrix \( A_{\text{LPT}} \) given in Bertsimas et al. (1999) by the prices of the stocks (to convert dollar impact into the absolute impact) and by the factor 100 (to convert percentage impact into linear). To make the QP
and QCQP problems convex programming problems, we consider diagonal impact matrix. From Bertsimas et al. (1999), we have impact matrix for considered stocks as follows:

\[
A_{\text{LPT}} = \begin{pmatrix}
12.4 & 0 & 0 & 0 & 0 \\
0 & 10.10 & 0 & 0 & 0 \\
0 & 0 & 14.4 & 0 & 0 \\
0 & 0 & 0 & 21.20 & 0 \\
0 & 0 & 0 & 0 & 11.70 \\
\end{pmatrix} \times 10^{-10}.
\]

By \(A_{\text{LPT}}\), we have following estimate of \(A\):

\[
A = \begin{pmatrix}
12.4 \times P_1 & 0 & 0 & 0 & 0 \\
0 & 10.10 \times P_2 & 0 & 0 & 0 \\
0 & 0 & 14.4 \times P_3 & 0 & 0 \\
0 & 0 & 0 & 21.20 \times P_4 & 0 \\
0 & 0 & 0 & 0 & 11.70 \times P_5 \\
\end{pmatrix} \times 10^{-8}
\]

\[
= \begin{pmatrix}
79.4375 & 0 & 0 & 0 & 0 \\
0 & 71.2050 & 0 & 0 & 0 \\
0 & 0 & 53.6400 & 0 & 0 \\
0 & 0 & 0 & 132.7650 & 0 \\
0 & 0 & 0 & 0 & 104.0569 \\
\end{pmatrix} \times 10^{-7}. \quad (50)
\]

5.3. Numerical example

The portfolio trading problem of five stocks listed in Table 2 is considered. We list the parameters discussed in Section 5.2 as follows:

\[
N = 5, \quad M = 1, \quad C = 0.0354, \quad \sigma_\eta = 0.9994, \quad X_1 = 0,
\]

\[
\Psi_\epsilon = \begin{pmatrix}
0.2368 & 0.0263 & 0.0225 & 0.0288 & 0.0998 \\
0.0263 & 0.0796 & 0.0080 & 0.0305 & 0.0476 \\
0.0225 & 0.0080 & 0.0279 & 0.0113 & 0.0325 \\
0.0288 & 0.0305 & 0.0113 & 0.0720 & 0.0502 \\
0.0998 & 0.0476 & 0.0325 & 0.0502 & 0.3678 \\
\end{pmatrix},
\]

\[
A = \begin{pmatrix}
79.4375 & 0 & 0 & 0 & 0 \\
0 & 71.2050 & 0 & 0 & 0 \\
0 & 0 & 53.6400 & 0 & 0 \\
0 & 0 & 0 & 132.7650 & 0 \\
0 & 0 & 0 & 0 & 104.0569 \\
\end{pmatrix} \times 10^{-7}, \quad B = \begin{pmatrix}
3.74 \times 10^{-5} \\
2.28 \times 10^{-5} \\
0.26 \times 10^{-5} \\
11.70 \times 10^{-5} \\
2.38 \times 10^{-5} \\
\end{pmatrix}. \quad (51)
\]

Apart from (51), the other parameters \(N, T_1, T_2, T_3, T_4, T_5, T, \bar{T}_1, \bar{T}_2, \bar{T}_3, \bar{T}_4, \bar{T}_5, U, \quad V, P_0, \) and \(P_{-1}, \) present in the QP and QCQP problems, are quoted at the required places. One trading time
Table 6
Optimal cost of execution corresponding to QP problem for different combination of the parameters \( N', P_0, P_{-1}, U, \) and \( V \)

<table>
<thead>
<tr>
<th>S. no.</th>
<th>( P_0 )</th>
<th>( P_{-1} )</th>
<th>AR(2) price model</th>
<th>AR(1) price model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>( U = [0.5 \ 0.5 \ 0.5 \ 0.5 \ 0.5] )</td>
<td>( U = [1 \ 1 \ 1 \ 1 \ 1] )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( V = [0.5 \ 0.5 \ 0.5 \ 0.5 \ 0.5] )</td>
<td>( V = [0 \ 0 \ 0 \ 0 \ 0] )</td>
</tr>
</tbody>
</table>

\( N' = 5 \) (all the assets are to be bought) optimal cost of purchasing \((\times 10^5)\) $

1. \( [64 \ 70 \ 37 \ 62 \ 88]' \) | \( [60 \ 68 \ 35 \ 60 \ 84]' \) | 317.3587 | 323.3815 |
2. \( [62 \ 69 \ 36 \ 61 \ 86]' \) | \( [62 \ 69 \ 36 \ 61 \ 86]' \) | 315.6097 | 316.3413 |
3. \( [60 \ 68 \ 35 \ 60 \ 84]' \) | \( [64 \ 70 \ 37 \ 62 \ 88]' \) | 313.1085 | 309.3026 |

\( N' = 3 \) (first three assets are to be bought) net optimal cost of purchasing and selling \((\times 10^5)\) $

4. \( [64 \ 70 \ 37 \ 62 \ 88]' \) | \( [60 \ 68 \ 35 \ 60 \ 84]' \) | 21.5079 | 23.3823 |
5. \( [62 \ 69 \ 36 \ 61 \ 86]' \) | \( [62 \ 69 \ 36 \ 61 \ 86]' \) | 21.6094 | 22.2176 |
6. \( [60 \ 68 \ 35 \ 60 \ 84]' \) | \( [64 \ 70 \ 37 \ 62 \ 88]' \) | 20.9237 | 21.0511 |

\( N' = 0 \) (all the assets are to be sold) optimal revenue of selling \((\times 10^5)\) $

7. \( [64 \ 70 \ 37 \ 62 \ 88]' \) | \( [60 \ 68 \ 35 \ 60 \ 84]' \) | 314.8895 | 318.6174 |
8. \( [62 \ 69 \ 36 \ 61 \ 86]' \) | \( [62 \ 69 \ 36 \ 61 \ 86]' \) | 312.3892 | 311.6174 |
9. \( [60 \ 68 \ 35 \ 60 \ 84]' \) | \( [64 \ 70 \ 37 \ 62 \ 88]' \) | 310.6977 | 304.0142 |

aThe other parameters are taken as \( T_1 = T_2 = T_3 = T_4 = T = 20 \) half-hourly trading periods, \( |S_1| = |S_2| = |S_3| = |S_4| = |S_5| = 10^5 \). The rest of the parameters of QP problem are as in (51).

period of half an hour is considered as in Bertsimas et al. (1999). We used MOSEK solver within MATLAB software to solve QP and QCQP problems by static approximation method.

Table 6 studies the optimal cost of execution for different combinations of parameters involved. \( P_0 \) and \( P_{-1} \) are considered for the cases when all the assets show upward (\( P_{n0} > P_{n(-1)} \)), rows labeled as 1, 4, and 7), no change (\( P_{n0} = P_{n(-1)} \), rows labeled as 2, 5, and 8), or downward movement (\( P_{n0} < P_{n(-1)} \), rows labeled as 3, 6, and 9) in their prices. Different choices of parameters \( U \) and \( V \) are based on the different autoregressive behavior of the assets of the portfolio. Columns 4 and 5, respectively, correspond to the AR(2) and AR(1) behavior in the price dynamics. Following are few observations that can be gauged from Table 6.

1. The trading strategy \( \{S_1, S_2, \ldots, S_T\} \) is a positive or negative vector, respectively, according to the buying or selling trading order. Accordingly, the impact (\( AS_t \)) on the execution price \( P_t \) is positive or negative, respectively, for a buying or selling trading order. Due to positive impact on prices the trade of buying increases the execution prices, whereas in case of selling the negative impact decreases the prices. Thus, for any combination of parameters \( P_0, P_{-1}, U, \) and \( V \), the cost of buying the portfolio is always greater than the revenue generated by selling it.
2. Attributing to general market behavior, the upward movement in prices always give high cost of buying or high revenue for any type of price path considered as compared to the case of no-change or downward movement in prices.
3. Consider the following relation:

\[
\min\{P_{n(t-1)}, P_{n(t-2)}\} \leq u_n P_{n(t-1)} + v_n P_{n(t-2)} \leq \max\{P_{n(t-1)}, P_{n(t-2)}\}.
\]

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Table 7
Optimal cost of execution corresponding to QP problem for different combination of the parameters $T_1, T_2, T_3, T_4, T_5, S_1, S_2, S_3, S_4,$ and $S_5^a$

<table>
<thead>
<tr>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$T_3$</th>
<th>$T_4$</th>
<th>$T_5$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
<th>$S_5$</th>
<th>Optimal execution cost ((\times 10^2)$ $$</th>
<th>Execution cost per share ((\times 10^4))</th>
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</thead>
<tbody>
<tr>
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<td>20</td>
<td>10</td>
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<td>10</td>
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<td>10</td>
<td>10</td>
<td>317.3587</td>
<td>63.4717</td>
</tr>
<tr>
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<td>15</td>
<td>15</td>
<td>15</td>
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<td>63.4784</td>
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<td>10</td>
<td>10</td>
<td>317.4664</td>
<td>63.4933</td>
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<td>2.5</td>
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<td>2.5</td>
<td>2.5</td>
<td>78.7723</td>
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<td>10</td>
<td>10</td>
<td>317.3587</td>
<td>63.4717</td>
</tr>
<tr>
<td>III</td>
<td>20</td>
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<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>317.4650</td>
<td>63.4930</td>
</tr>
</tbody>
</table>

$^a$Other parameters are considered as $N' = 5, \ P_0 = [64\ 70\ 37\ 62\ 88]', \ P_{-1} = [60\ 68\ 35\ 60\ 84]', \ U = [0.5\ 0.5\ 0.5\ 0.5\ 0.5]', \ V = [0.5\ 0.5\ 0.5\ 0.5\ 0.5'].$ The rest of the parameters are as in (51).

In view of relation (52), we have the following observations from Table 6.

(a) For the upward movement in prices \(P_{n(t-1)} > P_{n(t-2)}\), relation (52) gives

\[
u_n P_{n(t-1)} + v_n P_{n(t-2)} \leq P_{n(t-1)}.
\]

As under AR(2) price behavior, the price in any time period $t$ depends on the factor $u_n P_{n(t-1)} + v_n P_{n(t-2)}$, whereas in AR(1) price path the prices in $t$th time period depends on $P_{n(t-1)}$. Thus by inequality (53), the execution cost and revenue generated are less for AR(2) price model compared to AR(1) under upward movement in prices.

(b) For the no change in prices \(P_{n(t-1)} = P_{n(t-2)}\), we have

\[
u_n P_{n(t-1)} + v_n P_{n(t-2)} = P_{n(t-1)}.
\]

Thus by Equation (54), the AR(2) and AR(1) price dynamics show similar trends and thus optimal execution cost can show any behavior.

(c) In case of downward movement in prices \(P_{n(t-1)} < P_{n(t-2)}\), the relation (52) gives

\[
u_n P_{n(t-1)} + v_n P_{n(t-2)} \geq P_{n(t-1)}.
\]

By inequality (55), the no-impact prices under AR(2) dynamics is higher than that under AR(1) dynamics. Thus, the execution cost and revenue generated are more for AR(2) price model as compared to AR(1) under upward movement in prices.

Table 7 reflects the dependence of optimal execution cost corresponding to QP problem on the liquidities and the time of execution of the assets in the portfolio. The trading of buying all the assets
of portfolio is considered. The last two columns give optimal execution cost and execution cost per share. Table 7 is divided into three row subpanels I, II, and III. The first row subpanel I considers the case of same time of execution and same liquidities for all the assets, with decreasing $T$ from first to third row. The longer trade duration gives higher opportunity to trade in more favorable execution prices, and thus optimal execution cost decreases as $T$ increases. The second row subpanel, labeled as II, presents the effect of order size $S_n$ on optimal execution cost. The decrease in $S_n$ reduces the price impact and thus execution price decreases. This, in turn, reduces execution cost of trading. The third row subpanel III depicts the effect of difference in time of execution ($T_n$) or liquidity ($S_n$) of assets on optimal execution cost. Difference in time of execution results in higher execution cost, whereas increase or decrease in execution cost with liquidity difference depends on different situations. If assets with high execution prices have high liquidities, then the corresponding optimal execution cost would be high, and vice versa.

In Fig. 1, we compare the “right tail average realization of execution cost” corresponding to the naive strategy $N$, given by (29), with that of the optimal execution strategy $S$ of the QCQP problem. By equivalence relation of SSD, given by (28), we get

$$-\text{Cost}_S \succeq_{\text{SSD}} -\text{Cost}_N \text{ iff } \frac{1}{j} \sum_{i=1}^{j} \text{Cost}_S^{K+1-i} \leq \frac{1}{j} \sum_{i=1}^{j} \text{Cost}_N^{K+1-i} \forall j = 1, 2, \ldots, K,$$

where $\text{Cost}_S^1 \leq \text{Cost}_S^2 \leq \cdots \leq \text{Cost}_S^K$ and $\text{Cost}_N^1 \leq \text{Cost}_N^2 \leq \cdots \leq \text{Cost}_N^K$.

In Fig. 1, “right tail average realization of execution cost” for optimal strategy $S$ and naive strategy $N$, that is, $\frac{1}{j} \sum_{i=1}^{j} \text{Cost}_S^{K+1-i}$ and $\frac{1}{j} \sum_{i=1}^{j} \text{Cost}_N^{K+1-i}$, respectively, corresponding to five simulation points are compared. In order to gauge the robustness of the solution of QCQP problem against the different realizations of random vectors $\epsilon$ and $\eta$, we present two sets of solutions (Fig. 1—solid and dotted lines) $S$ and $N$. The two solutions correspond to the two different set of realizations of $\epsilon$ and $\eta$. It is evident from Fig. 1 that these solutions show some level robustness. The points on the solid and dotted lines corresponding to five realizations either coincide or differ by an small amount. Corresponding to the solid line realization, the expected execution cost with respect to $S$ and $N$ are $240.4442 \times 10^5$ $\$ and $248.6749 \times 10^5$, $\$ respectively. Thus, the QCQP problem results in an optimal trading strategy $S$, which has less expected execution cost corresponding to the reference strategy $N$, and at the same time dominates $\text{Cost}_N$ in SSD sense.

6. Conclusion and future work

The tremendous growth in the portfolio management industry and trading market has increased the size, volume, and speed of trading significantly. Thus, it is quite important to execute the trading order efficiently to reduce the price impact and thus minimize cost of execution. The efficient modeling of execution price path is also an important aspect of the optimal trading problem. In this paper, the optimal portfolio trading problem under AR(2) portfolio price path is discussed. The AR(2) portfolio price model is proposed as an extension of the AR(1) portfolio price path of Bertsimas et al. (1999). The optimal portfolio trading problem is modeled for the general case in which time of execution and liquidity of assets in the portfolio are allowed to be different. Moreover,
the mixed trade of buying and selling of assets in the portfolio are considered. For a risk-averse trader, the SSD-constraint is imposed in the problem formulation. The problem formulation for risk-neutral and risk-averse traders results in QP and QCQP problems, respectively. It is shown that under certain assumptions on the execution price path, the portfolio execution cost is a convex function of decision variables. This, in turn, makes QP and QCQP problems as convex programming problems that are computationally efficient to solve. The algorithm to solve QP and QCQP problems by the static approximation method is given. To gauge the dependence of optimal execution cost on the parameters of the AR(2) portfolio price path and the SSD-constraint a numerical example of trading in portfolio of five assets is provided.
On the basis of the work of Rudoy and Rohrs (2008) and Rudoy (2009), this paper can be further extended for the case of portfolio trading where the no-impact portfolio price path evolves according to a cointegrated vector autoregressive process.

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References

Appendix A

By definitions of $L_0$ and $L_1$ given by Equations (12) and (13), respectively, we have

$$L_0 = I_T, \quad UL_0 = L_1,$$

which proves (16) of Lemma 1.

Next, Equation (17) of Lemma 1 is proved. Consider the case when $t$ is an even positive integer less than $T$. That is, let $t = 2l$ for some positive integer $l$.

$$UL_t + VL_{t-1} = UL_{2l} + VL_{2l-1}$$

(by Equations (12) and (13))

$$= U\left[ \sum_{k=0}^{l} \binom{2l-k}{k} U^{2l-2k} V^k \right] + V\left[ \sum_{k=0}^{l-1} \binom{2l-1-k}{k} U^{2l-1-2k} V^k \right]$$

$$= \left[ \sum_{k=0}^{l} \binom{2l-k}{k} U^{2l+1-2k} V^k \right] + \left[ \sum_{k=0}^{l-1} \binom{2l-1-k}{k} U^{2l-1-2k} V^{k+1} \right]$$

$$= \left[ \sum_{k=0}^{l} \binom{2l-k}{k} U^{2l+1-2k} V^k \right] + \left[ \sum_{k=1}^{l} \binom{2l-k}{k-1} U^{2l+1-2k} V^k \right]$$
\[
\begin{align*}
&= \left[ U^{2l+1} + \sum_{k=1}^{l} \binom{2l-k}{k} U^{2l+1-2k} V^k \right] + \left[ \sum_{k=1}^{l} \binom{2l-k}{k-1} U^{2l+1-2k} V^k \right] \\
&= U^{2l+1} + \sum_{k=1}^{l} \left[ \binom{2l-k}{k} + \binom{2l-k}{k-1} \right] U^{2l+1-2k} V^k \\
&= U^{2l+1} + \sum_{k=1}^{l} \binom{2l+1-k}{k} U^{2l+1-2k} V^k \\
&= \sum_{k=0}^{l} \binom{2l+1-k}{k} U^{2l+1-2k} V^k \\
&= L_{2l+1} \\
&= L_{t+1}.
\end{align*}
\]

Similarly, Equation (17) can be proved for the case when \( t \) is an odd positive integer. The element-wise comparison in matrix Equations (16) and (17) implies the proof of (18) and (19), respectively.

Appendix B

We use the method of mathematical induction to prove Equation (20).

Considering Equation (11) for \( t = 2 \), we get \( X_2 = CX_1 + \eta_2 \).

Thus (20) is true for \( t = 2 \).

Let us suppose (20) is true for \( t = m \) for some positive integer \( m < T \). Next, we prove that (20) is true for \( t = m + 1 \).

By Equation (11) for \( t = m + 1 \),

\[
X_{m+1} = CX_m + \eta_{m+1}
\]

(By the assumption that (20) is true for \( t = m \).)

\[
= C \left( C^{m-1}X_1 + \eta_m + C\eta_{m-1} + \ldots + C^{m-2}\eta_2 \right) + \eta_{m+1}
\]

\[
= C^mX_1 + \eta_{m+1} + C\eta_m + \ldots + C^{m-1}\eta_2.
\]

Thus, by method of mathematical induction, Lemma 2 is proved.

Appendix C

We use the method of mathematical induction to prove Equation (21) of Lemma 3.
Considering Equation (10) for \( t = 2 \), we get

\[
    P_2 = UP_1 + VP_0 + AS_2 + BX_2 + \epsilon_2
\]

(by Equation (10) for \( t = 1 \))

\[
    = U(UP_0 + VP_{-1} + AS_1 + BX_1 + \epsilon_1) + VP_0 + AS_2 + BX_2 + \epsilon_2
\]

\[
    = (U^2 + V)P_0 + UVP_{-1} + (BX_2 + UBX_1) + (AS_2 + UAS_1) + (\epsilon_2 + U\epsilon_1)
\]

(by Equation (11) for \( t = 2 \))

\[
    = (U^2 + V)P_0 + UVP_{-1} + (B(CX_1 + \eta_2) + UBX_1) + (AS_2 + UAS_1) + (\epsilon_2 + U\epsilon_1) + B\eta_2
\]

(by Equations (12) and (13) for \( t = 1 \), we get \( L_2 \) and \( L_1 \), respectively; also \( L_0 = I_T \).

\[
    = L_2P_0 + VL_1P_{-1} + (L_0BC + L_1B)X_1 + (L_0AS_2 + L_1AS_1) + (L_0\epsilon_2 + L_1\epsilon_1) + B\eta_2.
\]

which proves (21) for \( t = 2 \).

Let us suppose (21) is true for all \( 2 < t \leq m \) for some positive integer \( m < T \). Next, we prove that (21) is true for \( t = m + 1 \).

By Equation (10) for \( t = m + 1 \), we get

\[
    P_{m+1} = UP_m + VP_{m-1} + AS_{m+1} + BX_{m+1} + \epsilon_{m+1}
\]

(by the assumption that Equation (21) is true for all \( 2 < t \leq m \), in particular for \( t = m \) and \( t = m - 1 \))

\[
    = U \left[ L_mP_0 + VL_{m-1}P_{-1} + \left( \sum_{k=0}^{m-1} L_kBC^{m-1-k} \right)X_1 + \sum_{k=0}^{m-1} L_kAS_{m-k} + \sum_{k=0}^{m-1} L_k\epsilon_{m-k} \right]
\]

\[
    + \sum_{k=0}^{m-2} \left( \sum_{r=0}^{k} L_rBC^{k-r} \right)\eta_{m-k} \right] + V \left[ L_{m-1}P_0 + VL_{m-2}P_{-1} + \left( \sum_{k=0}^{m-2} L_kBC^{m-2-k} \right)X_1 \right]
\]

\[
    + \sum_{k=0}^{m-2} L_kAS_{m-1-k} + \sum_{k=0}^{m-2} L_k\epsilon_{m-1-k} + \sum_{k=0}^{m-3} \left( \sum_{r=0}^{k} L_rBC^{k-r} \right)\eta_{m-1-k} \right]
\]

\[
    + AS_{m+1} + BX_{m+1} + \epsilon_{m+1}
\]

\[
    = (UL_m + VL_{m-1})P_0 + V \left( UL_{m-1} + VL_{m-2} \right)P_{-1} + (UL_0BC^{m-1})
\]

\[
    + \sum_{k=1}^{m-1} (UL_k + VL_{k-1})BC^{m-1-k}X_1 + \left( UL_0AS_m + \sum_{k=1}^{m-1} (UL_k + VL_{k-1})AS_{m-k} \right)
\]

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We introduce the following symbols that are required in the proof of Lemma 4.

\[ Z : \text{set of integers} \]

Appendix D

We introduce the following symbols that are required in the proof of Lemma 4.

\[ Z : \text{set of integers} \]
$\mathbb{C}$: set of complex numbers

$\text{Real}(z)$: real part of a complex number $z$

$\mathbb{D}$: unit disc in the complex plane, that is, $\{z \in \mathbb{C}; |z| < 1\}$

$H_+$: right half plane of Cartesian plane, that is, $\{z \in \mathbb{C}; \text{Re}(z) \geq 0\}$.

Consider the following:

$$2Q_n = \begin{pmatrix}
2L_{n0} & L_{n1} & L_{n2} & \cdots & L_{n(T-2)} & L_{n(T-1)} \\
L_{n1} & 2L_{n0} & L_{n1} & \cdots & L_{n(T-3)} & L_{n(T-2)} \\
L_{n2} & L_{n1} & 2L_{n0} & \cdots & L_{n(T-4)} & L_{n(T-3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
L_{n(T-1)} & L_{n(T-2)} & L_{n(T-3)} & \cdots & L_{n1} & 2L_{n0}
\end{pmatrix}$$

$$= \begin{pmatrix}
L_{n0} & 0 & 0 & \cdots & 0 & 0 \\
0 & L_{n0} & 0 & \cdots & 0 & 0 \\
0 & 0 & L_{n0} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & L_{n0}
\end{pmatrix}
+ \begin{pmatrix}
L_{n0} & L_{n1} & L_{n2} & \cdots & L_{n(T-2)} & L_{n(T-1)} \\
L_{n1} & L_{n0} & L_{n1} & \cdots & L_{n(T-3)} & L_{n(T-2)} \\
L_{n2} & L_{n1} & L_{n0} & \cdots & L_{n(T-4)} & L_{n(T-3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
L_{n(T-1)} & L_{n(T-2)} & L_{n(T-3)} & \cdots & L_{n1} & L_{n0}
\end{pmatrix}$$

$$= \tilde{Q} + \tilde{\tilde{Q}}.$$ \hspace{1cm} (A.2)

The matrix $\tilde{Q}$, being a diagonal matrix with positive entries (by relation (18), $L_{n0} = 1$), is a positive definite matrix.

Next, we prove that the matrix $\tilde{\tilde{Q}}$ is a nonnegative definite matrix.

Consider the following theorem of Korányi and Pukánszky (1963), which forms basis for the proof.

Korányi-Pukánszky Theorem: If the power series $\sum_{\alpha \in \mathbb{N}^l} a_{\alpha} z^\alpha$ represents a holomorphic function $f$ on the polydisc $\mathbb{D}^l$, then $\text{Real}(f(z)) \geq 0$ for all $z \in \mathbb{D}^l$ if and only if the map $\phi: \mathbb{Z}^l \to \mathbb{C}$ defined by

$$\phi(\alpha) = \begin{cases}
2\text{Real}(a_\alpha) & \text{if } \alpha = 0 \\
a_\alpha & \text{if } \alpha > 0 \\
a_{-\alpha} & \text{if } \alpha < 0 \\
0 & \text{otherwise}
\end{cases}$$

is positive, that is, the $k \times k$ matrix $[\phi(m_i - m_j)]$ is nonnegative definite for every choice of $m_1, m_2, \ldots, m_k \in \mathbb{Z}^n$.

We use the fact that the range of a holomorphic function given in the above theorem can be considered as the unit disc $\mathbb{D}$. For more details, refer to Gupta (2015, Chapter 4).
Consider the following function $f$ from $\mathbb{D}$ to $\mathbb{D}$:

$$f(z) = u_n z + v_n z^2,$$

(A.3)

where $u_n$ and $v_n$ are as in the dynamics of execution price path of the $n$th asset in the portfolio given by (9).

Consider Cayley map $\chi$ from $\mathbb{D}$ to $H_+$ as follows:

$$\chi(z) = \frac{1 + z}{1 - z}.$$  

(A.4)

The composition function $\chi o f$ from $\mathbb{D}$ to $H_+$ is given by

$$\chi o f(z) = \frac{1 + f(z)}{1 - f(z)} = \frac{2}{1 - f(z)} - 1.$$  

(A.5)

Let the power series expansion of the holomorphic function $\frac{1}{1 - f(z)}$ in $\mathbb{D}$ is as follows:

$$\frac{1}{1 - f(z)} = \sum_{p=0}^{\infty} c_p z^p.$$  

(A.6)

By Equations (A.5) and (A.6), we have

$$\frac{1 + f(z)}{1 - f(z)} = 2 \sum_{p=0}^{\infty} c_p z^p - 1.$$

By Equations (A.5) and (A.6), we have

$$\Rightarrow \quad (1 + f(z))\left[\frac{1}{1 - f(z)}\right] = 2 \sum_{p=0}^{\infty} c_p z^p - 1,$$

(by Equation (A.6))

$$\Rightarrow \quad (1 + u_n z + v_n z^2) \left[\sum_{p=0}^{\infty} c_p z^p\right] = 2 \sum_{p=0}^{\infty} c_p z^p - 1,$$

(by Equation (A.3))

$$\Rightarrow \quad u_n \left(\sum_{p=0}^{\infty} c_p z^{p+1}\right) + v_n \left(\sum_{p=0}^{\infty} c_p z^{p+2}\right) = \sum_{p=0}^{\infty} c_p z^p - 1.$$  

(A.7)

Comparing coefficients of different powers of $z$ from both sides of Equation (A.7), we get

$$c_0 = 1, \quad u_n c_0 = c_1.$$  

(A.8)
\[ u_n c_p + v_n c_{p-1} = c_{p+1}, \quad \forall \ p = 1, 2, \ldots \]  
\[ \text{(A.9)} \]

By set of Equations (18)–(19) and (A.8)–(A.9), we conclude
\[ c_p = L_{np} \quad \forall \ p = 0, 1, 2, \ldots, T. \]  
\[ \text{(A.10)} \]

Next we apply the Korányi-Pukánszky theorem to the function \( \chi \) of from \( D \) to \( H_+ \), with its power series expansion as \( 2 \sum_{p=0}^{\infty} c_p z^p - 1 \left( = 1 + 2 \sum_{p=1}^{\infty} c_p z^p \right) \), and the following choice of \( T \times T \) matrix:
\[
\left[ \phi(m_i - m_j) \right]_{i,j=1,2,\ldots,T} = \left[ \phi(i - j) \right]_{i,j=1,2,\ldots,T} \quad (m_i = i)
\]
\[
= \begin{pmatrix}
\phi(0) & \phi(-1) & \phi(-2) & \cdots & \phi(1 - T) \\
\phi(1) & \phi(0) & \phi(-1) & \cdots & \phi(2 - T) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\phi(T - 1) & \phi(T - 2) & \phi(T - 3) & \cdots & \phi(0)
\end{pmatrix}
\]
\[
= \begin{pmatrix}
2 & 2c_1 & 2c_2 & \cdots & 2c_{T-1} \\
2c_1 & 2 & 2c_1 & \cdots & 2c_{T-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
2c_{T-1} & 2c_{T-2} & 2c_{T-3} & \cdots & 2
\end{pmatrix}
\]
\[
= 2 \tilde{Q} \quad \text{(by (A.10)).}
\]

Thus by the Korányi-Pukánszky theorem the matrix \( 2 \tilde{Q} \), and hence \( \tilde{Q} \) is nonnegative definite.

The matrix \( \hat{Q} + \tilde{Q} \), being the sum of positive definite and nonnegative definite matrices, is positive definite. By Equation (A.2), \( 2Q_n \) is positive definite.

Thus matrix \( Q_n \), being half of a positive definite matrix \( 2Q_n \), is positive definite.

Hence, Lemma 4 is proved. \[ \blacksquare \]