Computers and Mathematics with Applications 60 (2010) 2433–2444

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A fluid queue modulated by two independent birth–death processes

Viswanathan Arunachalam [a], Vandana Gupta [b], S. Dharmaraja [b,*]

[a] Department of Mathematics, University of Los Andes, Bogota, Colombia

[b] Department of Mathematics, IIT Delhi, Hauz Khas, New Delhi, 110016, India

ARTICLE INFO

Article history:
Received 15 September 2009
Received in revised form 1 July 2010
Accepted 12 August 2010

Keywords:
Fluid queue
Markov process
Buffer occupancy distribution
IEEE 802.11 wireless networks
Performance measures

ABSTRACT

We present a fluid queue model driven by two independent finite state birth–death processes with the objective to study the buffer occupancy distribution in any intermediate node in a communication network. In a communication network, at any node, the arrival and service of the packets are with variable rates. To model this scenario we develop a fluid queue with an infinite capacity buffer which receives fluid at variable rate and also releases fluid at variable rates. Because of variable inflow and outflow rates of the fluid, the proposed fluid queue is driven by the current states of two independent finite state birth–death processes evolving in the background which on merging give rise to a continuous time Markov chain which is not a birth–death process. Using the fluid queue model, we obtain the steady-state distribution of the buffer occupancy at any intermediate node during packet transmission in a communication network. As a special case, we consider a wireless network based on the IEEE 802.11 standard. We present the buffer occupancy distribution at any intermediate node in closed form with a numerical illustration. Along with buffer occupancy distribution, we also obtain various performance measures such as expected buffer content, average throughput, server utilization and mean delay which are relevant to packet transmission in such a communication network. Finally, we present numerical results to illustrate the feasibility of the proposed model. The results are in accordance with the expected behavior of these performance measures.

1. Introduction

In communication networks, information (in the form of data packets) generated by a source node are delivered to their destination by routing them via a multiplexer, a switch, an information processor, or in general, a sequence of intermediate nodes. The information arriving at the intermediate node is buffered for service (transmission), the server typically being a communication channel or processing unit. In high-speed networks, the traffic is very bursty in nature. This bursty nature of the traffic in high-speed networks requires an understanding of steady-state behavior of the system. The steady-state analysis of the buffer content is useful in studying congestion in high-speed networks [1].

In this paper, we use a fluid queue modeling approach to study the buffer occupancy distribution in high-speed networks. Fluid models are a natural choice for problems involving continuous flow. For certain queueing systems where the flow consists of discrete entities, and the behavior of individuals is not important to identify the performance analysis, fluid queue models are useful as approximate models. In high capacity communication networks, the concept of fluid is based on the assumption that most important dynamics depend not on how individual packets are processed, but rather on how aggregates of packets are processed [2]. The applicability of these ideas is based on the fact that the packet size is a very small fraction of typical buffer capacities in the network. Hence, this modeling approach of fluid queues treats the real

* Corresponding author. Tel.: +91 11 26597104; fax: +91 11 26581005.
E-mail addresses: aviswana@uniandes.edu.co (V. Arunachalam), vandana_iitd@yahoo.com (V. Gupta), dharmar@maths.iitd.ac.in (S. Dharmaraja).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2010.08.039

information flow as a continuous stream rather than considering its discrete nature. Typically, the fluid represents the information stored in a buffer and waiting for transmission in a network. In the fluid queue models, the arrival and service processes are modulated by a random external environment, and the object of interest is to study the behavior of the buffer level in the long run.

Fluid queues have been widely used in the performance evaluation of high-speed communication networks [3]. Most of the classical research on stochastic fluid models in the area of telecommunications is based on the work of Anick, Mitra and Sondhi [4,5]. Stochastic fluid models for queues have been extensively studied in [6]. van Doorn and Scheinhardt have analyzed the steady-state behavior of fluid queues which are driven by an infinite-state birth–death process (BDP) [7,8]. Guillemin and Fabrice have discussed the stationary distribution of a fluid queue driven by a finite state Markov chain [9]. In [10], Lenin and Parthasarathy have studied numerically the behavior of fluid buffer driven by truncated BDP with general birth and death rates. A majority of the works aimed in obtaining the buffer occupancy distribution have considered Markov modulated fluid queues wherein the rate of information arriving and leaving the switching component is modulated according to the current state of an underlying Markov process. In [11–13], the authors have used the analogous nature of Quasi BDPs with fluid queues, and the matrix analytic approach of Quasi BDPs to obtain the steady-state distribution of the buffer level.

With the fluid queue approach for communication networks, the actual flow of information in a large number of small data packets is modeled as fluid. Following the same approach, we model the flow of information from one node to another via any intermediate node in a network. The objective is to obtain the steady-state distribution of the buffer content at any intermediate node which can give important information on the congestion in the network. The stored information at any intermediate node forms the fluid buffer. The information that arrives at any intermediate node has randomly varying arrival rates which depends on the feedback from other intermediate nodes. The service rates (packet transmission rates) are dependent on the transmission rates of the communication channel. Hence, to model this scenario, we consider a fluid queue which is driven by two different finite state BDPs. We consider an infinite capacity buffer in which the inflow is determined by one BDP and the outflow is determined by another BDP. Note that the two BDPs are independent of each other. We then merge the two background BDPs to form a continuous time Markov chain (CTMC). As a consequence, the considered fluid queue is driven by a single background CTMC (which is a not a BDP). This is a step ahead of the existing literature as in most of the literature on fluid queues, the net inflow rate of fluid into the buffer is determined by a single background BDP [8,10,14–18].

Fluid queue models of this type find applications in communication networks based on the IEEE 802.11 standard. In this paper, we obtain the steady-state distribution of the buffer content at any intermediate node in such a network. In addition to this, we also obtain performance measures like expected buffer content, average throughput, server utilization and mean delay relevant to any communication network.

The rest of the paper is organized as follows. Section 2 gives a description of the fluid model. Section 3 presents the application based analysis of the fluid queue model giving the steady-state distribution of the buffer occupancy and various performance measures. Section 4 gives the numerical illustration for the proposed model. Finally, Section 5 concludes the paper with some observations.

2. Model description

We consider a Markov modulated fluid queue with infinite buffer capacity. We assume that the buffer is building up and getting depleted with variable rates. To model the variable rate of inflow into the buffer, the inflow rate is determined by a BDP $\{\tilde{X}(t), t \geq 0\}$ with finite state space $\{1, 2, \ldots, N\}$. Let $\tilde{\lambda}_i, i = 1, 2, \ldots, N-1$ be the birth rates and $\tilde{\mu}_i, i = 2, 3, \ldots, N$ be the death rates of this BDP. When $\tilde{X}(t)$ is in some state $i,\ i \in \{1, 2, \ldots, N\}$, then the inflow rate into the fluid buffer is given by $\tilde{c}_i$, which can take any real value. When the buffer level reaches zero and the inflow rate at that time is negative, then the buffer level remains at zero until the inflow rate becomes positive.

To model the variable rate of outflow from the buffer, the outflow rate is determined by the states of another independent BDP $\{\tilde{Y}(t), t \geq 0\}$ with M states, $1, 2, 3, \ldots, M$. Let $\tilde{\alpha}_i, i = 1, 2, \ldots, M-1$ be the birth rates and $\tilde{\beta}_i, i = 2, 3, \ldots, M$ be the death rates of this BDP. When $\tilde{Y}(t)$ is in some state $i, i \in \{1, 2, 3, \ldots, M\}$, then the outflow rate from the fluid buffer is given by $\tilde{h}_i, i = \{1, 2, \ldots, M\}$.

By combining the above mentioned independent BDPs, we obtain a CTMC with finite number of states. We denote this CTMC by $\{\tilde{Z}(t), t \geq 0\}$ with state space $S = \{(1, 1), (1, 2), \ldots, (1, M), (2, 1), (2, 2), \ldots, (2, M), \ldots, (N, 1), (N, 2), \ldots, (N, M)\}$. This CTMC has NM states. Hence, we have a fluid queue driven by a CTMC (which is a not a BDP). In the next section, we present the analysis of this fluid queue in the context of a practical application.

3. Application based model analysis

The IEEE 802.11 wireless LAN (WLAN) is the most widely used WLAN standard nowadays [19]. Hence, we have modeled the flow of information from one node to another (via any intermediate node) in a network based on the IEEE 802.11 protocol using this fluid queue approach. The information is buffered at the intermediate node for service where the server typically is a communication channel. The rate at which the information arrives at an intermediate node fluctuates randomly [20]. This

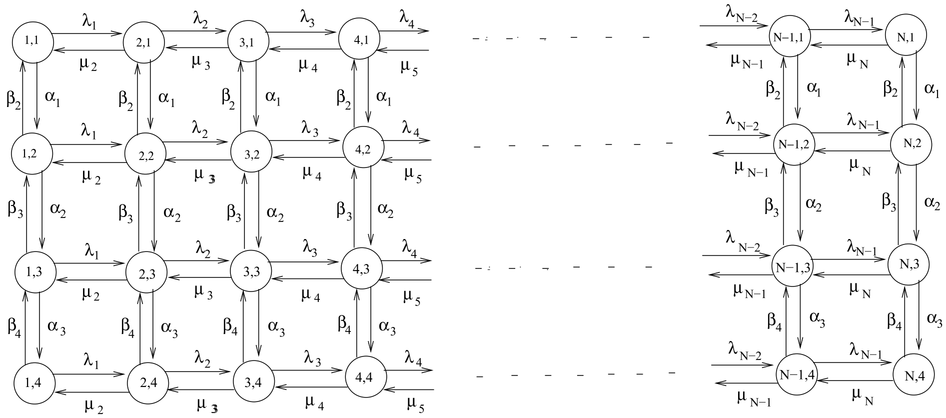

Fig. 1. State transition diagram of the background CTMC $Y(t)$.

fluctuation arises because sources do not maintain a constant data rate as it is dynamically regulated based on the feedback from other intermediate nodes. As a consequence, the inflow rates of fluid to the buffer is not constant and depends on the current state of a BDP evolving in the background. Let us assume that the inflow rate is determined by the BDP $\{X(t), t \geq 0\}$ with finite state space $\{1, 2, \ldots, N\}$. Let $\lambda_i, i = 1, 2, \ldots, N-1$ be the birth rates and $\mu_i, i = 2, 3, \ldots, N$ be the death rates of this BDP. When $X(t)$ is in some state $i, i \in \{1, 2, \ldots, N\}$, then the inflow rate into the fluid buffer is given by c_i, which can take any real value.

The release rate of fluid from the buffer depends on the transmission rate of the serving communication channel. The IEEE 802.11 WLAN standard supports multiple transmission rates that can maximize the system throughput in the face of adverse conditions. The IEEE 802.11b physical layer (PHY) specifies four different data rates, 1, 2, 5.5, and 11 Mbps [21]. In the current WLAN fields, different auto rate control algorithms have been proposed to specify how to change the rates according to channel conditions. But the ARF (Auto Rate Fallback) scheme [22] is the most popular auto rate control algorithm in IEEE 802.11b based WLAN products today. In the ARF scheme, ideally, users connect at the full 11 Mbps rate initially. But the transmission rate is downgraded to the next lower rate when the transmission continually fails and as a result the ACK (acknowledgement) from the receiver is consecutively missed. The transmission rate is upgraded back to the next higher rate if either the next consecutive transmissions are successful or some amount of time has passed. Based on the multiple transmission rates of communication channels and the ARF algorithm, let us assume that the outflow rate from the buffer is determined by the current state of another independent BDP $\{Y(t), t \geq 0\}$ with four states, 1, 2, 3, 4 evolving in the background. These four states represent the four different transmission rates supported by the IEEE 802.11 b protocol. The ith state of the process represents the current transmission rate of the communication channel. Let $\alpha_i, i = 1, 2, 3$ be the birth rates and $\beta_i, i = 2, 3, 4$ be the death rates of this BDP. When $Y(t)$ is in some state $i, i \in \{1, 2, 3, 4\}$, the outflow rate from the fluid buffer is given by h_i. Note that $h_1 > h_2 > h_3 > h_4$. The transitions of $\{Y(t), t \geq 0\}$ from one state to another represent the switching of the transmission rates from one step higher to one step lower or vice versa according to the ARF scheme.

As discussed in Section 2, on combining the BDPs $X(t)$ and $Y(t)$, we obtain a CTMC $\{Z(t), t \geq 0\}$ with state space $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), \ldots, (N, 1), (N, 2), (N, 3), (N, 4)\}$. This CTMC has $4N$ states. The state transition diagram for this process $\{Z(t), t \geq 0\}$ is shown in Fig. 1. Note that, in this CTMC, we have assumed that diagonal transitions are not feasible in a small time interval. Whenever, $Z(t) = (i, j), i \in \{1, 2, \ldots, N\}, j \in \{1, 2, 3, 4\}$, the outflow rate from the buffer is h_j. The considered fluid model driven by the underlying CTMC with four different outflow rates and state dependent inflow rates is shown in Fig. 2.

3.1. Buffer occupancy distribution

In this section, we obtain the steady-state distribution of the buffer occupancy. First, we describe the background stochastic process and follow it up with the description of the governing equation for the fluid model. Let

$$p_{i,j}(t) = \Pr\,[\text{The state of the CTMC } \{Z(t), t \geq 0\} \text{ is } (i, j) \text{ at the time } t]; \quad i = 1, 2, \ldots, N, j = 1, 2,\ t \geq 0.$$

For simplification, we enumerate the state (i, j) under a single index. Let $q_1(t), q_2(t), \ldots, q_{4N-1}(t), q_{4N}(t)$ be the $4N$ state probabilities of the CTMC $\{Z(t), t \geq 0\}$ which are given by:

$$q_{4n+i}(t) = p_{n+1,i}(t), \quad n = 0, 1, \ldots, N-1,\ i = 1, 2, 3, 4.$$

Hence, corresponding to the new indexing of the states of $\{Z(t), t \geq 0\}$, we define a new CTMC, $\{K(t), t \geq 0\}$ with finite state space $S = \{1, 2, \ldots, 4N\}$. The state transition diagram for this process $\{K(t), t \geq 0\}$ is shown in Fig. 3.

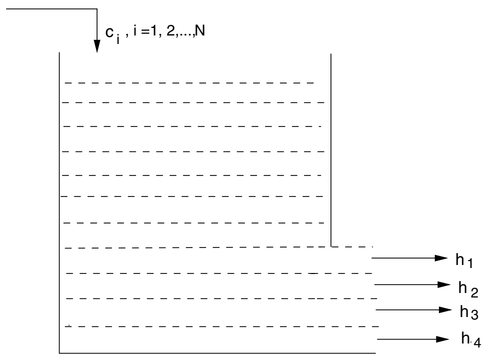

Fig. 2. Buffer Diagram for the fluid model.

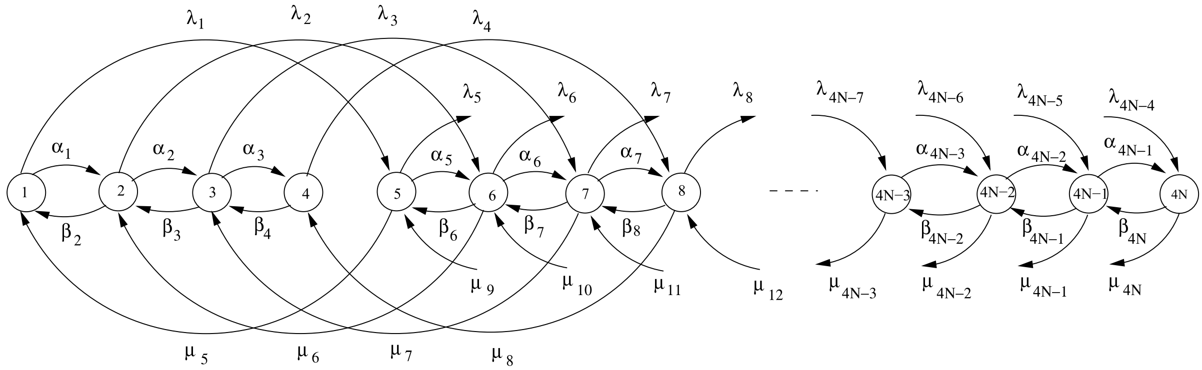

Fig. 3. State transition diagram of the CTMC $K(t)$.

As a consequence, the buffer content of the considered fluid queue is now determined by the CTMC $\{K(t), t \geq 0\}$. The infinitesimal generator of the CTMC $\{K(t), t \geq 0\}$ is given by

$$Q = \begin{pmatrix} \lambda_1 - \alpha_1 & \alpha_1 & 0 & 0 & \lambda_1 & \cdots \\ \beta_2 & -\alpha_2 - \beta_2 - \lambda_1 & \alpha_2 & 0 & 0 & \cdots \\ 0 & \beta_3 & -\alpha_3 - \beta_3 - \lambda_1 & \alpha_3 & 0 & \cdots \\ 0 & 0 & \beta_4 & -\beta_4 - \lambda_1 & 0 & \cdots \\ \mu_2 & 0 & 0 & 0 & -\alpha_1 - \mu_2 - \lambda_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1}$$

The buffer content at any time t is denoted by $C(t)$ and we assume that $C(0) = 0$. Hence, we now have a bi-dimensional stochastic process $\{C(t), K(t), t \geq 0\}$. The net flow rate into the buffer, denoted by r_i, is given by

For $n = 0, 1, 2, \ldots, N-1; j = n+1$

$$r_i = \begin{cases} c_j - h_1, & i = 4n+1 \\ c_j - h_2, & i = 4n+2 \\ c_j - h_3, & i = 4n+3 \\ c_j - h_4, & i = 4n+4. \end{cases} \tag{2}$$

We have the following differential equation [8]:

$$\frac{\mathrm{d}C(t)}{\mathrm{d}t} = \begin{cases} r_{K(t)}, & C(t) > 0 \\ 0, & C(t) = 0 \quad \text{and} \quad r_{K(t)} < 0. \end{cases}$$

This implies that whenever the CTMC $\{K(t), t \geq 0\}$ is in state i, $i \in S$, the corresponding net flow rate is r_i. Also each r_i must be either positive or negative with at least one $r_i > 0$ as otherwise the buffer will remain empty forever.

The buffer occupancy distribution is defined as

$$F_i(t, x) = \Pr\{K(t) = i, C(t) \leq x\}; \quad t \geq 0,\ x \geq 0,\ i \in S \tag{3}$$

where $F_i(t,x)$ is the probability that the background Markov process $\{K(t), t \geq 0\}$ is in some state i and the buffer content is less then or equal to some quantity x. The distribution of the buffer occupancy is a mixed distribution with a positive mass at $x = 0$, given as

$$F_i(t,x) = \begin{cases} 0; & \text{for } x < 0, t \geq 0, i \in S \\ \Pr\{K(t) = i, C(t) = 0\}; & \text{for } x = 0, t \geq 0, i \in S \\ \Pr\{K(t) = i, C(t) \leq x\}; & \text{for } x > 0, t \geq 0, i \in S. \end{cases} \tag{4}$$

In the long run, as $t \to \infty$ and $x \to \infty$,

$$\sum_{i=0}^{4N} F_i(t,x) = 1. \tag{5}$$

The governing differential equation for the fluid queue is given by [17]

$$\frac{\partial F_i(t,x)}{\partial t} = -r_i \frac{\partial F_i(t,x)}{\partial x} + \sum_{j \in S} F_j(t,x) Q(j,i), \quad i \in S \tag{6}$$

where $F_i(t,x)$ is defined in Eq. (3) and Q is given in Eq. (1).

In the next subsection, we obtain the steady-state distribution for the buffer occupancy.

3.2. Steady-state distribution of the buffer occupancy

We define the steady-state distribution (in the long run as $t \to \infty$) as

$$F_i(x) = \lim_{t \to \infty} F_i(t,x) \tag{7}$$

where $F_i(t,x)$ is defined in Eq. (3). For the steady-state solution to exist, we need a stability condition. For the fluid queue, this stability condition is that as $t \to \infty$, the stationary net flow rate should be negative, that is

$$\sum_{i=1}^{4N} q_i \quad r_i < 0. \tag{8}$$

Now, using (7), in the long run as $t \to \infty$, Eq. (6) becomes

$$0 = -r_i \frac{\mathrm{d}F_i(x)}{\mathrm{d}x} + \sum_{j \in S} F_j(x) Q(j,i), \quad i \in S. \tag{9}$$

Let $\vec{F}(x)$ be column vector formed by the $4N$ stationary probabilities and is given by

$$\vec{F}(x) = (F_1(x), F_2(x), \ldots, F_{4N}(x))^T$$

and let R be the diagonal matrix given by

$$R = \mathrm{diag}(r_1, r_2, \ldots, r_{4N}).$$

Hence, the Eq. (9) in matrix form is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} \vec{F}(x) = R^{-1} Q^T \vec{F}(x). \tag{10}$$

In order to obtain the steady-state distribution of the buffer occupancy, we need to solve Eq. (10), for which we use the following results.

3.3. Some preliminary results

Lemma 1 (*Steady-state solution of* $\boldsymbol{\{K(t), t \geq 0\}}$). *Let q_i, $i = 1, 2, \ldots, 4N$ be the steady-state probabilities of the CTMC $\{K(t), t \geq 0\}$ with state space $S = \{1, 2, \ldots, 4N\}$. In steady-state, the net inflow rate to any state is equal to the net outflow rate from that state.*

Note that for numerical simplification, we have assumed that

$$\lambda_i = \lambda, \quad \text{for } i = 1, 2, \ldots, N-1$$
$$\mu_i = \mu, \quad \text{for } i = 2, \ldots, N$$
$$\alpha_i = \alpha, \quad \text{for } i = 1, 2, 3$$
$$\beta_i = \beta, \quad \text{for } i = 2, 3, 4.$$

Hence, the steady-state balance equations of the CTMC $\{K(t), t \geq 0\}$ are given as:

For states 1, 2, 3 *and* 4:

$$\begin{cases} (\lambda+\alpha)q_1 - \beta q_2 - \mu q_5 & = & 0 \\ -\alpha q_1 + (\alpha+\beta+\lambda)q_2 - \beta q_3 - \mu q_6 & = & 0 \\ -\alpha q_2 + (\alpha+\beta+\lambda)q_3 - \beta q_4 - \mu q_7 & = & 0 \\ -\alpha q_3 + (\beta+\lambda)q_4 - \mu q_8 & = & 0. \end{cases}$$

For $i = \{1, 2, \ldots, N-2\}$:

$$\begin{cases} \underline{\textit{For states } (5, 9, 13, \ldots, 4N-7):} \\ \qquad -\lambda q_{4i-3} + (\alpha+\lambda+\mu)q_{4i+1} - \beta q_{4i+2} - \mu q_{4i+5} = 0 \\ \underline{\textit{For states } (6, 10, 14, \ldots, 4N-6):} \\ \qquad -\lambda q_{4i-2} - \alpha q_{4i+1} + (\alpha+\beta+\lambda+\mu)q_{4i+2} - \beta q_{4i+3} - \mu q_{4i+6} = 0 \\ \underline{\textit{For states } (7, 11, 15 \ldots, 4N-5):} \\ \qquad -\lambda q_{4i-1} - \alpha q_{4i+2} + (\alpha+\beta+\lambda+\mu)q_{4i+3} - \beta q_{4i+4} - \mu q_{4i+7} = 0 \\ \underline{\textit{For states } (8, 12, 16, \ldots, 4N-4):} \\ \qquad -\lambda q_{4i} - \alpha q_{4i+3} + (\beta+\lambda+\mu)q_{4i+4} - \mu q_{4i+8} = 0 \end{cases}$$

and,

For states $4N-3$, $4N-2$, $4N-1$ *and* $4N$:

$$\begin{cases} -\lambda q_{4N-7} + (\alpha+\mu)q_{4N-3} - \beta q_{4N-2} = 0 \\ -\lambda q_{4N-6} - \alpha q_{4N-3} + (\alpha+\beta+\mu)q_{4N-2} - \beta q_{4N-1} = 0 \\ -\lambda q_{4N-5} - \alpha q_{4N-2} + (\alpha+\beta+\mu)q_{4N-1} - \beta q_{4N} = 0 \\ -\lambda q_{4N-4} - \alpha q_{4N-1} + (\beta+\mu)q_{4N} = 0. \end{cases}$$

From above equations, we get

$$q_5 = \frac{(\lambda+\alpha)q_1 - \beta q_2}{\mu}$$
$$q_6 = \frac{-\alpha q_1 + (\alpha+\beta+\lambda)q_2 - \beta q_3}{\mu}$$
$$q_7 = \frac{-\alpha q_2 + (\alpha+\beta+\lambda)q_3 - \beta q_4}{\mu}$$
$$q_8 = \frac{-\alpha q_3 + (\beta+\lambda)q_4}{\mu}.$$

Similarly, for $i = \{2, \ldots, N-2\}$

$$q_{4i+1} = \frac{-\lambda q_{4i-3} + (\alpha+\lambda+\mu)q_{4i+1} - \beta q_{4i+2}}{\mu}$$
$$q_{4i+2} = \frac{-\lambda q_{4i-2} - \alpha q_{4i+1} + (\alpha+\beta+\lambda+\mu)q_{4i+2} - \beta q_{4i+3}}{\mu}$$
$$q_{4i+3} = \frac{-\lambda q_{4i-1} - \alpha q_{4i+2} + (\alpha+\beta+\lambda+\mu)q_{4i+3} - \beta q_{4i+4}}{\mu}$$
$$q_{4i+4} = \frac{-\lambda q_{4i} - \alpha q_{4i+3} + (\beta+\lambda+\mu)q_{4i+4}}{\mu}.$$

From above equations, it can be observed that all the $q_i's$, $5 \leq i \leq 4N$, can be written in terms of q_1, q_2, q_3 and q_4 by the back substitution method. Hence, once we have the values of q_1, q_2, q_3 and q_4, we can obtain all the other $q_i's$, $5 \leq i \leq 4N$.

And to get the values of q_1, q_2, q_3 and q_4, we have the following conditions:

$$\sum_{i=1}^{4N} q_i = 1. \tag{11}$$

In addition, from the states of different outflow rates of the four state BDP $\{Z(t), t \geq 0\}$, we have

$$\left.\begin{aligned}
&\alpha\sum_{i=0}^{N-1} q_{4i+1} = \beta\sum_{i=0}^{N-1} q_{4i+2}\\
&(\alpha+\beta)\sum_{i=0}^{N-1} q_{4i+2} = \alpha\sum_{i=0}^{N-1} q_{4i+1} + \beta\sum_{i=0}^{N-1} q_{4i+3}\\
&(\alpha+\beta)\sum_{i=0}^{N-1} q_{4i+3} = \alpha\sum_{i=0}^{N-1} q_{4i+2} + \beta\sum_{i=0}^{N-1} q_{4i+4}\\
&\alpha\sum_{i=0}^{N-1} q_{4i+3} = \beta\sum_{i=0}^{N-1} q_{4i+4}
\end{aligned}\right\}. \tag{12}$$

Using the Eqs. (11) and (12), q_1, q_2, q_3 and q_4 can be obtained.

Lemma 2. *The following results hold for the matrix $R^{-1}Q^T$ where R is given by Eq. (2) and Q is given by Eq. (1)*

1. *The matrix has $4N$ real eigen values.*
2. *One of the eigen values is zero.*
3. *The number of negative eigen values is equal to the number of states of $K(t)$ with positive net flow.*

Refer [5,20] for the proof.

Theorem 1. *The steady-state distribution of buffer occupancy is given by*

$$\vec{F}(x) = \vec{q} + \sum_{j=1}^{d_+} k_j e^{z_j x}\Phi_j$$

where $\vec{F} = (F_1, F_2, \dots, F_{4N})^T$ and $\vec{q}$ is the column vector of steady-state probabilities of the CTMC $\{K(t), t \geq 0\}$ model. That is, $\vec{q} = (q_1, q_2, \dots, q_{4N})^T$ and $z_1, z_2, \dots, z_{4N}$ are the $4N$ eigen values of $R^{-1}Q^T$ with respective eigen vectors $\Phi_1, \Phi_2, \dots, \Phi_{4N}$. d^+ is the number of states of the CTMC $\{K(t), t \geq 0\}$ with positive net flow rate and k_j's are some constants.

Proof. Using Lemma 2 for the matrix $R^{-1}Q^T$, the solution of the system of differential equations given by Eq. (10) is

$$\vec{F}(x) = \sum_{i=1}^{4N} k_i e^{z_i x}\Phi_i. \tag{13}$$

As $\vec{F}(x)$ is a vector of probabilities and its components are bounded, therefore the coefficients corresponding to the negative eigen values must be zero. Also, in order to solve Eq. (10), we need a stability condition and the boundary conditions. The stability condition is that the stationary net flow rate should be negative which is given in Eq. (8). The stability condition assures the existence of the stationary probabilities. And the boundary conditions are given by:

$$\begin{aligned}
&F_i(0) = 0, \quad \text{for state } i \text{ with positive flow}\\
&F_i(\infty) = q_i, \quad \forall\, i. \quad \square
\end{aligned}$$

By Lemma 2, the number of negative eigen values is equal to the number of states with positive net flow. Let the number of states with positive net flow be d_+. Define $z_1, z_2, \dots, z_{d_+}$ as the d_+ negative eigen values with its respective eigen vectors $\Phi_1, \Phi_2, \dots, \Phi_{d_+}$. Also, by Lemma 2, $z_0 = 0$ is a eigen value of $R^{-1}\,Q^T$. Hence, Eq. (13) can be written as:

$$\vec{F}(x) = k_0\Phi_0 + \sum_{j=1}^{d_+} k_j e^{z_j x}\Phi_j.$$

By the boundary condition, as $x \to \infty$, for $i = 1, 2, \dots, 4N$,

$$q_i = F_i(\infty) = k_0\,\Phi_0^{(i)}$$

which implies,

$$\vec{F}(x) = \vec{q} + \sum_{j=1}^{d_+} k_j e^{z_j x}\Phi_j. \tag{14}$$

Hence the theorem.

To find the numerical solutions, we need to determine:

$$\vec{q}, k_1, k_2, \dots, k_{d_+}, z_1, z_2, \dots, z_{d_+}, \Phi_1, \Phi_2, \dots, \Phi_{d_+}.$$

In order to find the constants $k_1, k_2, \ldots, k_{d_+}$, the boundary condition is used with $x = 0$ for state i with positive net flow:

$$0 = q_i + \sum_{j=1}^{d_+} k_j \Phi_j^{(i)}.$$

This is a linear system in $k_1, k_2, \ldots, k_{d_+}$ with d_+ equations that can be represented by the known quantities $\vec{q}, z_1, z_2, \ldots, z_{d_+}$ and $\Phi_1, \Phi_2, \ldots, \Phi_{d+}$.

3.4. Performance measures

In this section, we define various performance measures related to the packet transmission in a network, such as average throughput, server utilization, expected buffer content and mean delay. We give the formulation for these measures in context of the fluid queue model.

1. *Average throughput:* For a communication network, system throughput is defined as the number of data packets transmitted per unit time. In other words, it can be defined as the number of packets departing from the system per unit time. Hence, the system throughput is defined as

$$\textit{Throughput} = \textit{Service rate} \times \textit{Probability that the system is non empty}.$$

From the state transition diagram of $\{Z(t), t \geq 0\}$ as shown in Fig. 3, it can be observed that we have four rows with service rates (which is equivalent to the release rates) h_1, h_2, h_3 and h_4, respectively. Hence, the system throughput can be given by

$$\begin{aligned}\text{Throughput} = {} & h_1 \times \Pr[K(t) = 4i+1; C(t) > 0] + h_2 \times \Pr[K(t) = 4i+2; C(t) > 0] \\ & + h_3 \times \Pr[K(t) = 4i+3; C(t) > 0] + h_4 \times \Pr[K(t) = 4i+4; C(t) > 0]; \\ & \text{as } t \to \infty, \quad \text{for } i = 0, 1, \ldots, N-1.\end{aligned}$$

Now, from Eqs. (4) and (5), as $t \to \infty$ and $x \to \infty$,

$$\begin{aligned}\Pr\{K(\infty) = i, C(\infty) > 0\} &= \Pr[K(\infty) = i, C(\infty) < \infty] - \Pr[K(\infty) = i, C(\infty) \leq 0] \\ &= F_i(\infty) - F_i(0).\end{aligned}$$

Hence, the throughput is given by

$$\begin{aligned}\text{Throughput} = {} & h_1 \times \sum_{i=0}^{N-1} [F_{4i+1}(\infty) - F_{4i+1}(0)] + h_2 \times \sum_{i=0}^{N-1} [F_{4i+2}(\infty) - F_{4i+2}(0)] \\ & + h_3 \times \sum_{i=0}^{N-1} [F_{4i+3}(\infty) - F_{4i+3}(0)] + h_4 \times \sum_{i=0}^{N-1} [F_{4i+4}(\infty) - F_{4i+4}(0)].\end{aligned} \tag{15}$$

2. *Server utilization:* It is defined as the fraction of time the server is busy. In other words, it can be defined as the probability that the system is non-empty. For the fluid queue model, it can be defined as the probability that the buffer is non empty. This can be obtained as:

$$\textit{Utilization} = \textit{Probability that buffer is non empty}.$$

From Eqs. (4) and (5), this is given as

$$\text{Utilization} = 1 - \sum_{i=1}^{4N} F_i(0). \tag{16}$$

3. *Expected buffer content:* It is defined as the average buffer content in the long run. Now, if there is a non-negative real valued random variable, say T, and expectation of T, $E(T) < \infty$, then

$$E(T) = \int_0^\infty \{1 - F(t)\} \mathrm{d}t.$$

Using the above formula for expectation, the expected buffer content (X) can be given as

$$E(X) = \int_0^\infty \left[1 - \sum_{i=1}^{4N} F_i(x)\right] \mathrm{d}x. \tag{17}$$

Table 1
List of parameters.

Rates	Meaning	Value
λ	Arrival rate of $X(t)$	0.02–0.06
μ	Departure rate of $X(t)$	0.06–0.09
α	Forward rate of $Y(t)$	0.03–0.05
β	Backward rate of $Y(t)$	0.03–0.05
c_i	Inflow rate of fluid into the buffer when $X(t)$ is in state i	$c_1 = 1.5$ Mbps, $c_2 = 2.75$ Mbps, $c_3 = 3$ Mbps, $c_4 = 4.5$ Mbps, $c_5 = 5$ Mbps, $c_6 = 6.5$ Mbps
h_j	Outflow rate of fluid into the buffer when $Y(t)$ is in state j	$h_1 = 11$ Mbps, $h_2 = 5.5$ Mbps, $h_3 = 2$ Mbps, $h_4 = 1$ Mbps

4. *Mean delay:* It is defined as the average time taken for the packet transmission. It includes the waiting time when the packet is in the buffer and the packet transmission time. Now, [23,24] implies that Little's law hold good for fluid queues also. Hence, using Little's law, average delay for our model can be obtained as:

$$\text{Mean delay} = \frac{\text{Expected buffer content}}{\text{Mean inflow rate}}. \tag{18}$$

In this fluid model, the inflow rate depends on the states of the background CTMC $\{K(t), t \geq 0\}$. Hence, we get the conditional mean delay given that the CTMC $\{K(t), t \geq 0\}$ is in some state i. The mean delay is then given by

$$\begin{aligned}\text{Mean delay} = {} & [\text{Expected buffer content} \mid \text{Background CTMC} \\ & \{K(t), t \geq 0\} \text{ is in states corresponding to rate } c_i]\Pr[\text{Background CTMC } \{K(t), t \geq 0\} \text{ is} \\ & \text{in states corresponding to rate } c_i]/c_i\end{aligned}$$

which is equal to

$$= \sum_{i=1}^{N} \left[\frac{\int_0^\infty \left[\sum_{j=i}^{4i} q_j - \sum_{j=i}^{4i} F_j(x) \right] \mathrm{d}x \times \sum_{j=i}^{4i} q_j}{c_i} \right]. \tag{19}$$

4. Numerical illustrations and observations

In this section, we present the numerical results obtained for the steady-state distribution of the buffer occupancy. Along with this, we also present the numerical results for average throughput, server utilization, expected buffer content and mean delay. For the purpose of numerical illustration, we have taken the number of states, $4N = 24$. The values of other parameters are given in Table 1. The parameter values are chosen only for the purpose of numerical illustration of the closed form results.

First we present the steady-state distribution of the buffer occupancy. The steady-state distribution of buffer occupancy is defined as

$$F(x) = \Pr[\text{buffer content } C(t) \leq x].$$

Hence, we have

$$F(x) = \sum_{i=1}^{4N} F_i(x)$$

where $F_i(x), i \in S$ are obtained by using Theorem 1.

Fig. 4 shows the variation of $F(x)$ with the buffer content x, where x is varied from 0 to 10 000. It can be seen from the graph that $F(x) \to 1$ as $x \to \infty$. Also, there is a positive mass at $x = 0$ because it may happen that when the background Markov process is in some state i at some time point $t \geq 0$, the buffer content is zero, i.e., $x = 0$. Hence, this shows that the buffer occupancy has mixed distribution.

Next, we obtain the performance measures mentioned in Section 3.4 using the definitions given in the same section. Fig. 5 shows the variation of average throughput with the channel switching rates α and β. It can be seen from the graph that the throughput increases with an increase in the rate α and decrease in the rate β.

Fig. 6 shows the variation of server utilization with the arrival rate λ of the background BDP $\{X(t), t \geq 0\}$. For this graph, other parameters are fixed as $\mu = 0.07, \alpha = 0.04$ and $\beta = 0.03$. As expected, it can be seen from the graph that utilization

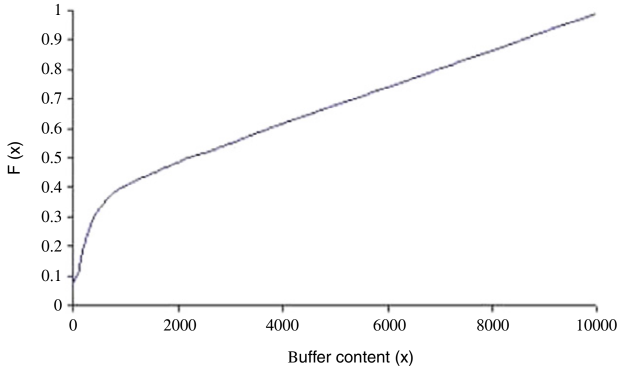

Fig. 4. Cumulative distribution function of buffer occupancy.

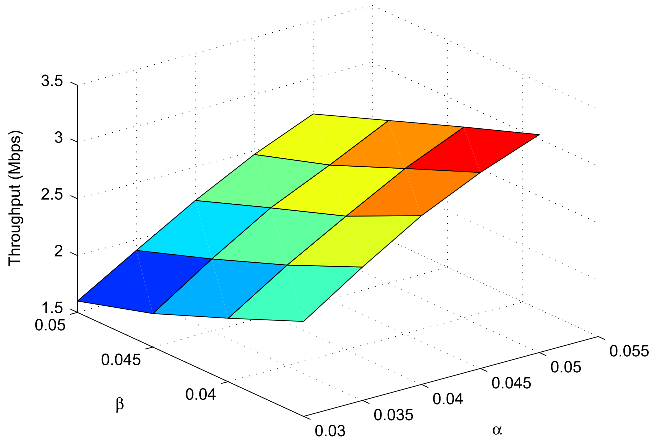

Fig. 5. Average throughput vs channel switching rates.

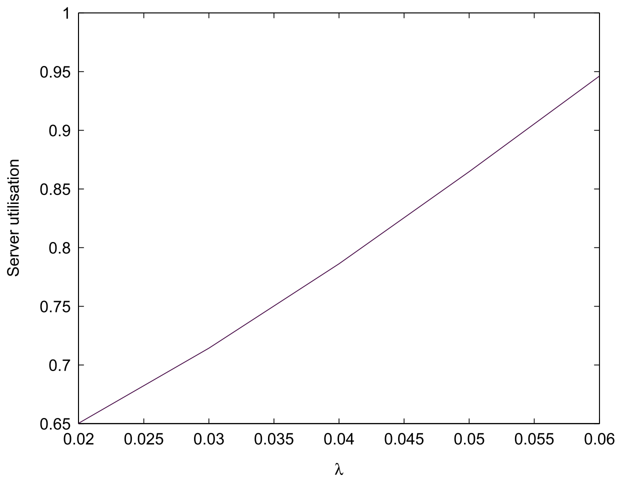

Fig. 6. Server utilization vs arrival rate λ.

increases with an increase in the arrival rate. This is so because as more arrivals take place, the buffer content will also increase, making the server more busy.

Fig. 7 shows the variation of expected buffer content with the arrival rate λ of the background BDP $\{X(t), t \geq 0\}$. For this graph also, other parameters are fixed as $\mu = 0.07$, $\alpha = 0.04$ and $\beta = 0.03$. The graph shows that the expected buffer content increases with an increase in the arrival rate, as predicted. This is because as more arrivals take place, the expected buffer content also increases.

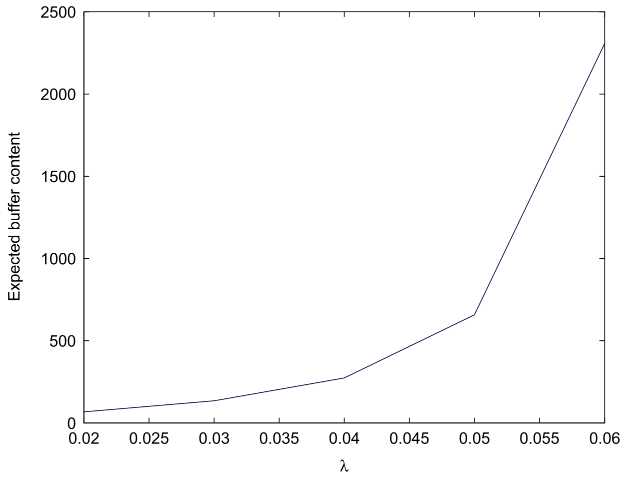

Fig. 7. Expected buffer content vs arrival rate λ.

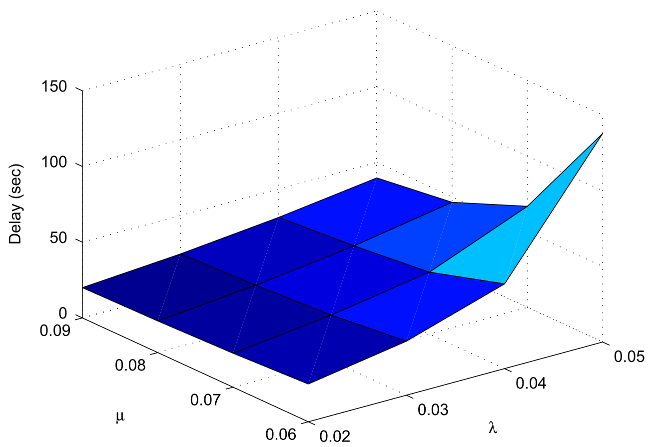

Fig. 8. Mean delay vs rates λ and μ.

Fig. 8 shows the variation of mean delay with the rates λ and μ. For this graph, other parameters are fixed as $\alpha = 0.04$ and $\beta = 0.03$. As foreseeable, the graph shows that delay increases with an increase in the arrival rate λ and a decrease in the service rate μ, and vice versa. This is because if the arrival rate is more and the service rate is less, the delay will be more.

Note that we have modeled only two aspects of the IEEE 802.11 protocol, i.e., variable arrival rate of information, and variable service rate based on different transmission rates of communication channel in our paper. As we have not taken into consideration the other important aspects of the protocol, we have not validated out results with the performance measures of the communication networks.

5. Conclusion and future work

In this paper, we develop a fluid queue model driven by two independent finite state BDPs. Most of the work done in the area of fluid queues involve a single BDP as the background Markov process in which the net flow rate of fluid into the buffer depends on the states of a single background BDP. We, on the other hand, present a fluid queue driven by two independent background BDPs in which the inflow rate is dependent on one BDP and the outflow rate is dependent on another independent BDP. The two background BDPs on merging give rise to a CTMC, which is a not a BDP. The goal of the paper is to study the steady-state buffer occupancy at any intermediate node in a communication network where the packets arrive with variable rate and is serviced with variable rate. As a special case, we consider a wireless network based on the IEEE 802.11 standard. Using the fluid queue model, we obtain the steady-state distribution of the buffer content at any intermediate node which gives information about congestion in such a network. We also obtain certain performance measures related to the packet transmission in a network, such as average throughput, server utilization, expected buffer content and mean delay at the fluid queue level. The results are in accordance with the expected behavior of these measures in a network.

The bursty nature of the traffic carried in high-speed networks requires an understanding of steady-state as well as transient behavior of the system. Where steady-state analysis is useful to study the congestion in networks, the transient analysis is of critical value in understanding the dynamical behavior of the system to control the congestion. Hence, as future work, we are planning to obtain transient distribution of the buffer content using the fluid queue approach.

Acknowledgements

The authors thank the referees for their helpful suggestions which led to an improvement of an earlier draft of this paper. V. Arunachalam gratefully acknowledges the support from the Faculty of Sciences, University of Los Andes, Bogota, Colombia. This research work is also supported by the Department of Science and Technology, India. One of the authors (V.G.) would like to thank CSIR, India for providing her financial support through a Senior Research Fellowship.

References

[1] R. Izmailov, Deterministic service of identical and independent on–off fluid sources, Stochastic Models 12 (2) (1996) 329–342.
[2] R. Roy, R.C. Mudumbai, S.S. Panwar, Analysis of TCP congestion control using a fluid model, in: Proc. ICC, 2001, pp. 2396–2403.
[3] S. Blaabjerg, H. Andersson, Approximating the heterogeneous fluid queue with a birth–death fluid queue, IEEE Transaction on Communications 43 (5) (1995) 1884–1887.
[4] D. Anick, D. Mitra, M.M. Sondhi, Stochastic theory of a data handling system with multiple sources, Bell System Technical Journal 61 (1982) 871–1984.
[5] D. Mitra, Stochastic theory of a fluid model of producers and consumers coupled by a buffer, Advanced Applied Probability 20 (1988) 646–676.
[6] V.G. Kulkarni, Fluid models for single buffer systems, in: Frontiers in Queueing: Models and Applications in Science and Engineering, CRC Press, USA, 1997, pp. 321–338.
[7] E.A. van Doorn, W.R.W. Scheinhardt, Analysis of birth–death fluid queues, in: Proc. KAIST Applied Mathematics Workshop, Korea, 1996, pp. 13–29.
[8] E.A. van Doorn, W.R.W. Scheinhardt, A fluid queue driven by an infinite-state birth–death process, in: V. Ramaswami, P.E. Wirth (Eds.), Proc. 15th International Teletraffic Congress, Elsevier, Amsterdam, 1997, pp. 465–475.
[9] Guillemin Fabrice, Stationary analysis of a fluid queue driven by some countable state space Markov chain, Methodology And Computing in Applied Probability 9 (2007) 521–540.
[10] R.B. Lenin, P.R. Parthsarathy, A computational approach for fluid queues driven by truncated birth–death processes, Methodology and Computing in Applied Probability 2 (2000) 373–392.
[11] V. Ramaswami, Matrix analytic methods for stochastic fluid flows, in: D. Smith, P. Hey (Eds.), Teletraffic Engineering in a Competitive World (Proceedings of the 16th International Teletraffic Congress), Elsevier Science B.V., Edinburgh, UK, 1999, pp. 1019–1030.
[12] S. Ahn, V. Ramaswami, Fluid flow models and queues a connection by stochastic coupling, Communications in Statistics-Stochastic Models 19 (3) (2003) 325–348.
[13] A. da Silva Soares, G. Latouche, Matrix-analytic methods for fluid queues with finite buffers, Performance Evaluation 63 (2006) 295–314.
[14] I. Adan, J. Resing, Simple analysis of a fluid queue driven by an M/M/1 queue, Queueing Systems 22 (1996) 171–174.
[15] N. Barbot, B. Sericola, Stationary solution to the fluid queue fed by an M/M/1 queue, Journal of Applied Probability 39 (2002) 359–369.
[16] B. Sericola, Exact transient solution of M/M/1 driven fluid queue, International Journal of Computer Mathematics 82 (6) (2005) 659–671.
[17] E.A. van Doorn, A.A. Jagers, J.S.J. de Wit, A fluid reservoir regulated by a birth–death process, Stochastic Models 4 (1988) 457–472.
[18] R. Malhotra, M.R.H. Mandjes, W.R.W. Scheinhardt, J.L. van den Berg, A feedback fluid queue with two congestion control thresholds, Mathematical Methods of Operations Research 70 (2009) 149–169.
[19] LAN MAN Standards Committee of the IEEE Computer Society, ANSI/IEEE Std. 802.11, Part 11: Wireless LAN Medium Access Control (MAC) and Physical Layer (PHY) Specifications, 1999 Edition.
[20] T.E. Stern, A.I. Elwalid, Analysis of separable Markov-modulated rate models for information-handling systems, Advanced Applied Probability 23 (1991) 105–139.
[21] N. Kim, S. Choi, H. Yoon, IEEE 802.11b WLAN Performance with Variable Transmission Rates: in View of High Level Throughput, in: International Conference on Networking, ICN, 2, 2005, pp. 1080–1087.
[22] Y.-J. Kim, Y.-J. Suh, An efficient rate switching scheme for IEEE 802.11 wireless LANs, in: Proc. 61st IEEE Vehicular Technology Conference, VTC, 4, 2005, pp. 2364–2368.
[23] O. Czerniak, U. Yechiali, Fluid polling systems, Queueing Systems 63 (2009) 401–435.
[24] Y. Liu, W.B. Gong, Perturbation analysis for stochastic fluid queueing systems, Discrete Event Dynamic Systems 12 (4) (2002) 391–416.