BIPARAMETRIC SURFACES

1. Bilinear surface
2. Synthetic Surfaces

PARAMETRIC REPRESENTATION

In parametric surfaces a vector valued function \( P(u,v) \) of two variables is used as follows:

\[
P(u,v) = [x \quad y \quad z] = [x(u,v) \quad y(u,v) \quad z(u,v)]
\]

\[u_{\text{min}} \leq u \leq u_{\text{max}}; \quad v_{\text{min}} \leq v \leq v_{\text{max}}\]

A surface may be one patch or constructed using several patches. All complex surfaces are represented using many patches.
SPHERICAL PATCH

\[ Q(\theta, \phi) = [\cos \theta \sin \phi \quad \sin \theta \sin \phi \quad \cos \phi] \]

**Parametric Derivatives**

The four patch boundaries are defined by 4 position vectors and 8 tangent vectors 2 at each corner. The tangent vectors can be obtained by finding the derivatives w.r.t \( \theta \) and \( \phi \).

\[
Q(\theta, \phi) = [\cos \theta \sin \phi \quad \sin \theta \sin \phi \quad \cos \phi]
\]

\[
Q_\theta(\theta, \phi) = [-\sin \theta \sin \phi \quad \cos \theta \sin \phi \quad 0]
\]

\[
Q_\phi(\theta, \phi) = [\cos \theta \cos \phi \quad \sin \theta \cos \phi \quad -\sin \phi]
\]

Also the shape of the interior near the 4 corners is influenced by 4 twist vectors defined as the cross derivatives of the parameters \( \theta \) and \( \phi \).

\[
Q_{\theta,\phi}(\theta, \phi) = \frac{\partial^2 Q}{\partial \theta \partial \phi} = \frac{\partial^2 Q}{\partial \phi \partial \theta} = [-\sin \theta \cos \phi \quad \cos \theta \cos \phi \quad 0]
\]
**Surface Normal**

The surface normal at any point is calculated as the cross product of the parametric derivatives.

\[ Q_\theta(\theta, \phi) \times Q_\phi(\theta, \phi) = \begin{vmatrix} i & j & k \\ -\sin \theta \sin \phi & \cos \theta \sin \phi & 0 \\ \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi \end{vmatrix} = \begin{bmatrix} -\cos \theta \sin^2 \phi & \sin \theta \sin^2 \phi & -\sin^2 \theta \sin \phi \cos \phi \end{bmatrix} \]

On a surface patch the isoparametric lines are orthogonal. Therefore the dot product of parametric derivatives is zero.

\[ Q_\theta(\theta, \phi) \cdot Q_\phi(\theta, \phi) = \begin{bmatrix} -\sin \theta \sin \phi & \cos \theta \sin \phi & 0 \end{bmatrix} \begin{bmatrix} \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi \end{bmatrix} = 0 \]

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**Surface Mapping**

Two parameter surfaces from the parametric (rectangular) space to three dimensional xyz object space.

\[ u = C_1, \ C_3 \leq w \leq C_4, \ x = x(u, w) \]
\[ u = C_2, \ C_3 \leq w \leq C_4, \ y = y(u, w) \]
\[ w = C_3, \ C_1 \leq u \leq C_2, \ z = z(u, w) \]
3D Surface Mapping

Example of 3D surface mapping - R&A Example 6.8.

\[ x = (u - w)^2 \]
\[ y = u - w^2 \]
\[ z = uw \]

![3D Surface Mapping Diagram](image)

Bilinear Surface

A bilinear patch or surface is constructed by the four corners of a unit square \( P(0,0), P(0,1), P(1,1) \) and \( P(1,0) \). Any point in the interior of the surface is obtained by interpolation between the two boundary curves.

\[ Q(u, w) = P(0,0)(1-u)(1-w) + P(0,1)(1-u)w + P(1,0)u(1-w) + P(1,1)uw \]

**Example 6.8 R&A**
**Ruled and Developable Surface**

A ruled surface is produced when a straight line (curve) is moved along a path with one degree of freedom.

**Test for Ruled Surface:** At any point on the surface rotate a plane containing the surface normal. If at least at one orientation all the points on the edge of the plane contact the surface, then the surface is ruled in that direction.

\[ Q(u, w) = P(u, 0)(1 - w) + P(u, 1)w \]

Alternately

\[ Q(u, w) = P(0, w)(1 - u) + P(1, w)u \]

**Surface Curvature:** Gaussian and Mean Curvatures
Developable Surfaces

Development of a surface is the process of unfolding it onto a planar face without stretching or tearing.

Not all ruled surfaces are developable. However, all developable surfaces are ruled.


The average and Gaussian curvature for a surface are defined respectively as:

\[
H = \frac{\kappa_{\text{min}} + \kappa_{\text{max}}}{2} = \frac{A|Q_u|^2 - 2BQ_u Q_v + C|Q_v|^2}{2|Q_u \times Q_v|^3}
\]

\[
K = \kappa_{\text{min}} \cdot \kappa_{\text{max}} = \frac{AC - B^2}{|Q_u \times Q_v|^4}
\]

Where

\[
\begin{bmatrix}
A & B & C
\end{bmatrix} = [Q_u \times Q_v][Q_{uu} \quad Q_{uw} \quad Q_{vw}]
\]

For a developable surface \( K = 0 \) everywhere

Surface Characteristics

A developable surface has the property that it can be made out of sheet metal, since such a surface must be obtainable by transformation from a plane (which has Gaussian curvature 0) and every point on such a surface lies on at least one straight line.

A surface of constant Gaussian curvature: It is a surface where \( K = \text{const} \).

Minimal Surface

Minimal surfaces are defined as surfaces with zero mean curvature \( H = 0 \). A minimal surface parametrized as \( \mathbf{X} = (u, v, \mathbf{h}(u, v)) \) therefore satisfies Lagrange's equation,

\[
(1 + \mathbf{h}_v^2) \mathbf{h}_{uu} - 2 \mathbf{h}_u \mathbf{h}_v \mathbf{h}_{uv} + (1 + \mathbf{h}_u^2) \mathbf{h}_{vv} = 0
\]

Complete Surface

A complete surface is a surface which has no edges.

Complete Minimal Surface

A surface which is simultaneously complete and minimal. There have been a large number of fundamental breakthroughs in the study of such
**Surface Characteristics**

**Complete Minimal Surface**
A surface which is simultaneously complete and minimal. There have been a large number of fundamental breakthroughs in the study of such surfaces in recent years, and they remain the focus of intensive current research.

**Beltrami’s Theorem**
Let \( f : M \rightarrow N \) be a surface mapping. If either \( M \) or \( N \) has constant curvature, then both surfaces have constant curvature.

**Smooth Surface**
A surface parameterized in variables \( u \) and \( v \) is called smooth if the tangent vectors in the \( u \) and \( v \) directions satisfy

\[
T_u \times T_v \neq 0
\]

---

**Helicoid: A minimal Surface**

The (circular) helicoid is the minimal surface having a (circular) helix as its boundary. It is the only ruled minimal surface other than the plane. The parametric equation of helicoid is:

\[
\begin{align*}
x &= u \cos \theta \\
y &= u \sin \theta \\
z &= c \theta
\end{align*}
\]
Shape and Curvature of Developable Surfaces

<table>
<thead>
<tr>
<th>$K_{\text{min}}, K_{\text{max}}$</th>
<th>$K$</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>Same sign</td>
<td>$&gt; 0$</td>
<td>Elliptic</td>
</tr>
<tr>
<td>Opp. sign</td>
<td>$&lt; 0$</td>
<td>Hyperbolic</td>
</tr>
<tr>
<td>One or both zero</td>
<td>0</td>
<td>Cylindrical / Conical</td>
</tr>
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Linear Coons Surface

If four boundary curves, $P(u,0)$, $P(u,1)$, $P(0,w)$ and $P(1,w)$ are known, and bilinear blending function is used for the interior of the surface, a linear Coons surface can be generated.

It appears that we can simply add two singly ruled interpolations to obtain this. This poses some problems at the corners of the patch, e.g. $P(0,0)$ will be added twice. To do the corrections some terms at the end are subtracted

Wrong

$$Q(u, w) = P(u,0)(1 - w) + P(u,1)w + P(0,w)(1 - u) + P(1,w)u$$

Right

$$Q(u, w) = P(u,0)(1 - w) + P(u,1)w + P(0,w)(1 - u) + P(1,w)u$$
$$- P(0,0)(1 - u)(1 - w) - P(0,1)(1 - u)w$$
$$- P(1,0)u(1 - w) - P(1,1)uw$$

Parametric space
Coons Bicubic Surface

If we use higher order polynomials for boundary curves and interior blending functions then we get a more flexible surfaces. Patches of surfaces are joined together to generate the required surface. Such surfaces are called sculptured surfaces.

Coons bicubic surface patch uses normalised cubic splines for all four boundary curves and also for blending functions to define the interior.

![Geometry for Coons patch]

$$P(t) = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P'_1 \\ P'_2 \end{bmatrix}$$

Coons Bicubic Surface

$$F_1(t) = 2t^3 - 3t^2 + 1 \quad 0 \leq t \leq 1$$
$$F_2(t) = -2t^3 + 3t^2$$
$$F_3(t) = t^3 - 2t^2 + 1$$
$$F_4(t) = t^3 - t^2$$

$$Q(u, w) = \begin{bmatrix} P(0, 0) & P(0, 1) & P_n(0, 0) & P_n(0, 1) \\ P(1, 0) & P(1, 1) & P_n(1, 0) & P_n(1, 1) \\ P_n(0, 0) & P_n(0, 1) & P_n(0, 0) & P_n(0, 1) \\ P_n(1, 0) & P_n(1, 1) & P_n(1, 0) & P_n(1, 1) \end{bmatrix} \begin{bmatrix} F_1(w) \\ F_2(w) \\ F_3(w) \\ F_4(w) \end{bmatrix}$$

$$Q(u, w) = \begin{bmatrix} u^3 & u^2 & u & 1 \\ w^3 & w^2 & w & 1 \end{bmatrix}$$
Derivatives of Coons Bicubic Surface

\[
Q_u(u, w) = [U'] [N] [P] [N]' [W]
\]

\[
Q_w(u, w) = [U] [N] [P] [N]' [W']
\]

\[
Q_{uw}(u, w) = [U'] [N] [P] [N]' [W']
\]

\[
Q_{uw}(u, w) = [U'] [N] [P] [N]' [W']
\]

\[
Q_{ww}(u, w) = [U] [N] [P] [N]' [W'']
\]

\[
[U] = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \quad [W] = \begin{bmatrix} w^3 & w^2 & w & 1 \end{bmatrix}
\]

\[
[U'] = \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix}
\]

\[
[W'] = \begin{bmatrix} 3w^2 & 2w & 1 & 0 \end{bmatrix}
\]

\[
[U''] = \begin{bmatrix} 6u & 2 & 0 & 0 \end{bmatrix}
\]

\[
[W''] = \begin{bmatrix} 6w & 2 & 0 & 0 \end{bmatrix}
\]

Surface normal is given by \( n = Q_u \times Q_w \)

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Coons Patch and Shape Control

We have 4 corner vertices, 8 tangent vectors and 4 twist vectors as input data. The tangent vectors and twist vector magnitude and directions can be changed to influence the shape.

Effect of changing the magnitude of tangent vectors: Consider the position vectors on \( y=0 \) plane. If the \( y \) magnitude of the tangent vectors are zero then the surface is flat. If it is not zero then it is bowed. As we increase the magnitude the surface bulges up and results in self intersection patch when the order of magnitude is increased further.

Geometry for Coons patch
Coons Patch and Shape Control

Effect of changing the direction of tangent vectors

When we change the direction (sign) of the tangent vectors at any vertex it changes the local convex portion of the patch to concave shape.

Effect of changing the TWIST vectors: We can control the interior shape of the surface by changing twist vectors keeping the tangent vectors at the corners the same. By modifying the twist vector (both magnitude and direction) we can influence the shape of the surface. Modifying one twist vector magnitude affects about a quarter portion near to that corner. Increasing the magnitude lifts up the surface further. However in contrast to what happens at higher orders of magnitude in the case of tangent vectors, the self intersecting surface does not result in this case.

The surface with zero twist vectors is called Ferguson or F patch.