Solid Modeling Techniques

Half Spaces

Boundary Representation (B-rep)

Half Spaces

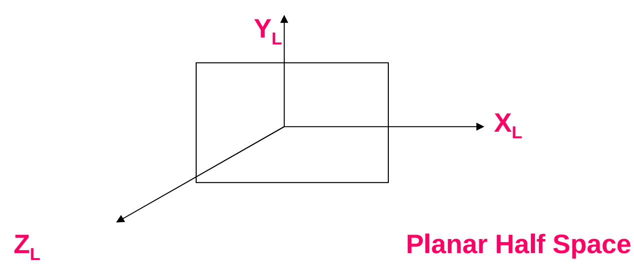
Half spaces form a basic representation scheme for bounded solids.

A half space is a regular point set in **E^3** and is given by:

$$H = \{P : P \in E^3 \ and \ f(P) < 0\}$$

A planar half space is represented as:

$$H = \{(x, y, z) : z < 0\}$$

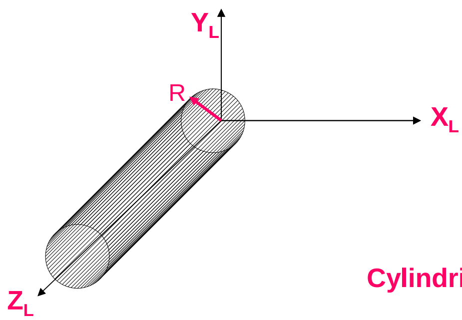


Classification: Unevaluated boundary based, spatial based

Half Spaces

A cylindrical Half Space is a given by:

$$H = \{(x, y, z) : x^2 + y^2 < R^2\}$$

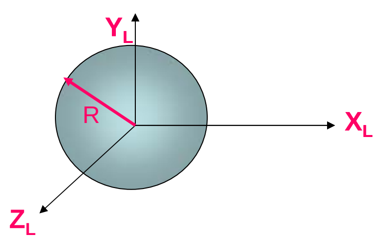


Cylindrical Half Space

Half Spaces

A spherical Half Space is a given by:

$$H = \{(x, y, z) : x^2 + y^2 + z^2 < R^2\}$$

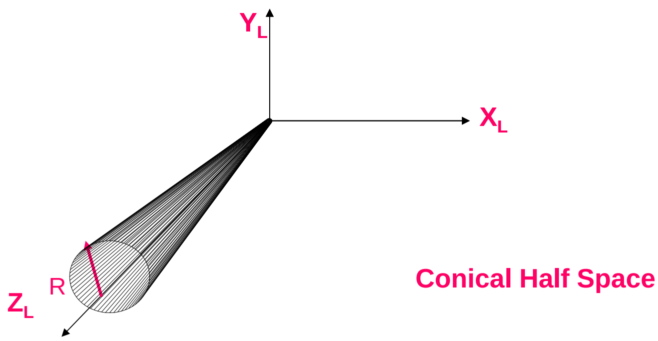


Spherical Half Space

Half Spaces

A conical Half Space is a given by:

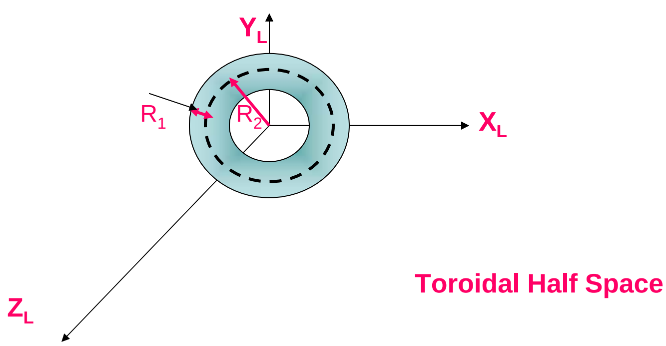
$$H = \{(x, y, z) : x^2 + y^2 < (\tan(\alpha / 2)z)^2\}$$



Half Spaces

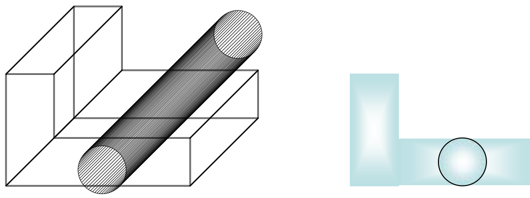
A toroidal Half Space is a given by:

$$H = \{(x, y, z) : (x^2 + y^2 + z^2 - R_2^2 - R_1^2 < 4R_2^2(R_1^2 - z^2)\}$$



Constructing Solids with Half Spaces

Complex objects can be modeled by combining Half Space using set operations



$$S = \bigcup \left(\bigcap_{i=0}^{n} H_i \right)$$

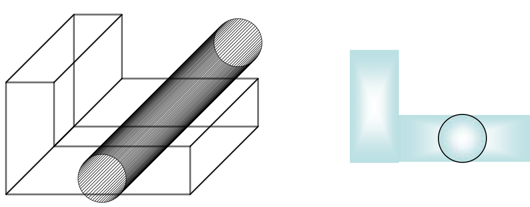
Constructing solids with Half Space

Advantages and Disadvantages of Half Spaces

Advantages:

The main advantage is its conciseness of representation compared to other modeling schemes.

It is the lowest level representation available for modeling a solid object



Disadvantages:

The representation can lead to unbounded solid models as it depend on user manipulation of half spaces

The modeling scheme is cumbersome for ordinary users / designers

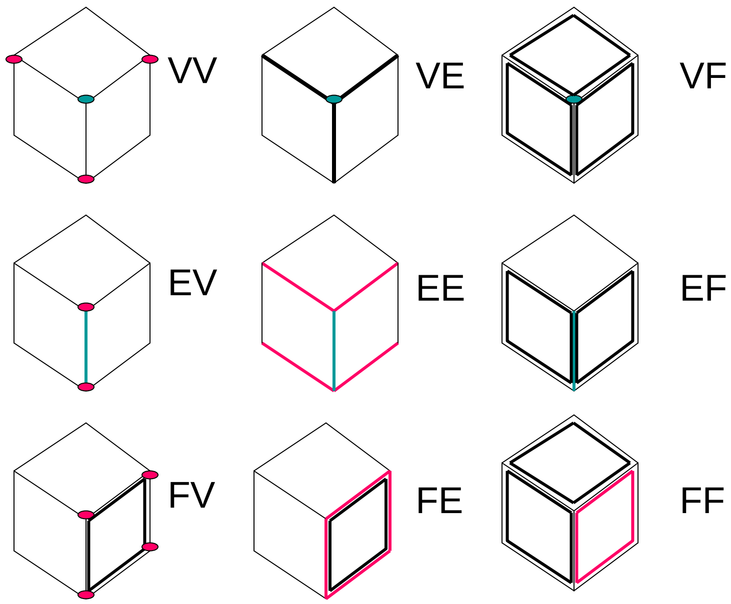
Boundary Representation (B-rep)

- **Closed Surface**: One that is continuous without breaks.
- **Orientable Surface** :One in which it is possible to distinguish two sides by using surface normals to point to the inside or outside of the solid under consideration.
- **Boundary Model** :Boundary model of an object is comprised of closed and orientable faces, edges and vertices. A database of a boundary model contains both its topology and geometry.
- **Topology** :Created by Euler operations
- **Geometry** :Includes coordinates of vertices, rigid motions and transformations

Boundary Representation (B-rep)

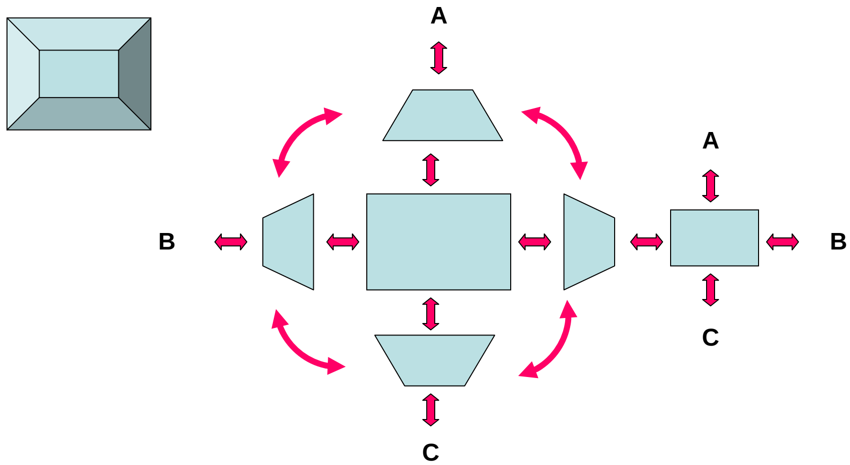
- Involves surfaces that are
  - *closed, oriented manifolds embedded* in 3-space
- A **manifold** surface:
  - each point is *homeomorphic* to a disc
- A manifold surface is **oriented** if:
  - any path on the manifold maintains the orientation of the normal
- An oriented manifold surface is **closed** if:
  - it partitions 3-space into points inside, on, and outside the surface
- A closed, oriented manifold is **embedded in 3-space** if:
  - Geometric (and not just topological) information is known

Adjacency Topology in B-rep

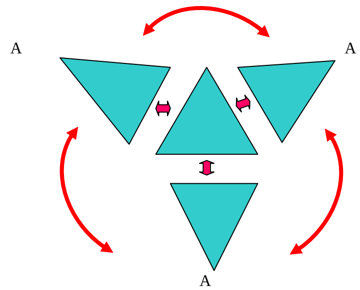


Topological Atlas and Orientability

- The simplest data structure keeps track of adjacent edges. Such a data structure is called an atlas.

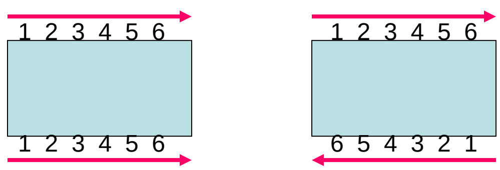


Topological Atlas of a Tetrahedron



Topological Atlas and Orientability

- The orientability indicated with arrows or numbers as shown below. We see that the orienation preserving arrows are in two opposite rotational directions i.e., clockwise and anticlockwise. While orienation reversing arrows are in the same rotational directions.

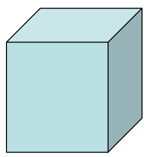


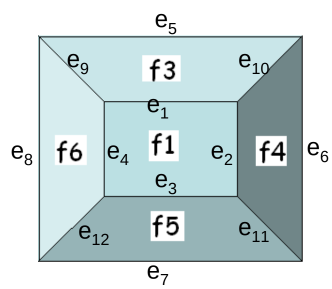
Orientation Preserving

Orientation Reversing

Schlegel Diagrams

- A common form of embedding graphs on planar faces is called Schlegel Diagram. It is a projection of its combinatorial equivalent of the vertices, edges and faces of the embedded boundary graph on to its surface. Here the edges may not cross except at their incident vertices and vertices may not coincide.

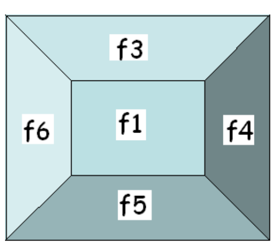


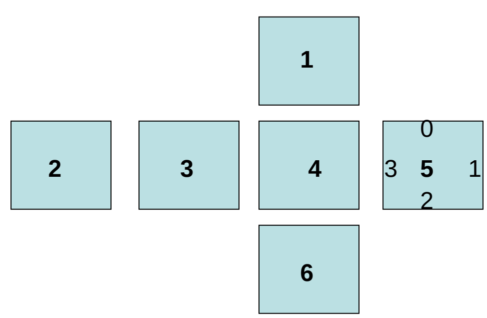


Schlegel Diagram of a Cube

Atlas of Cube

- An atlas of a cube can also be given by the arrangement of its faces as shown below

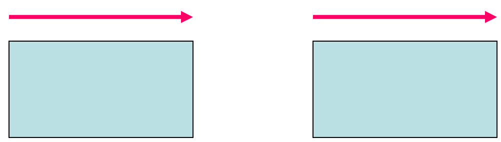




Atlas of Cube

Some Examples of Atlases

- While orienation reversing arrows are in the same rotational directions.

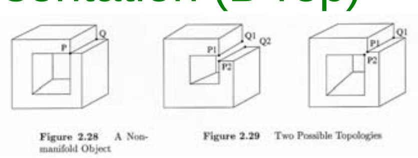


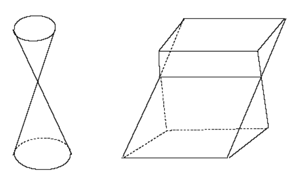
Cylinder **Mobius Strip**

Can you think of the atlases of Torus ? Klein bottle ?

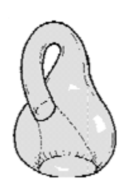
Boundary Representation (B-rep)

- Non-manifold surfaces
- Non-oriented Manifolds





Moebius strip



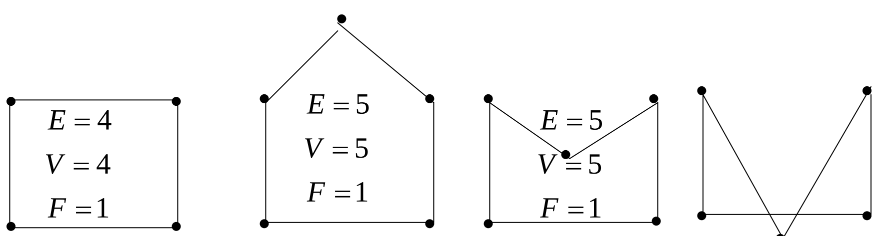
Klein bottle

Object Modeling with B-rep

Both polyhedra and curved objects can be modeled using the following primitives

- **Vertex**: A unique point (ordered triplet) in space.
- **Edge** :A finite, non-selfintersecting directed space curve bounded by two vertices that are not necessarily distinct.
- **Face** :Finite, connected, non-selfintersecting region of a closed, orientable surface bounded by one or more **loops**.
- **Loop** :An ordered alternating sequence of vertices and edges. A loop defines non-self intersecting piecewise closed space curve which may be a boundary of a face.
- **Body** :An independent solid. Sometimes called a shell has a set of faces that bound single connected closed volume. A minimum body is a point (vortex) which topologically has one face one vortex and no edges. A point is therefore called a seminal or singular body.
- **Genus** :Hole or handle.

Boundary Representation



Original object **Modified objects** **Nonsense object**

- **Euler Operations (Euler –Poincare‘ Law):**
  The validity of resulting solids is ensured via Euler operations which can be built into CAD/CAM systems.
- **Volumetric Property calculation in B-rep:**
  It is possible to compute volumetric properties such as mass properties (assuming uniform density) by virtue of Gauss divergence theorem which converts volume integrals to surface integrals.

Leonhard Euler (1707 – 1783)
Henri Poincaré (1854 – 1912)

Euler-Poincare Law

- **Euler (1752)** a Swiss mathematician proved that polyhedra that are homomorphic to a sphere are topologically valid if they satisfy the equation:

$F - E + V - L = 2(B - G)$	General
$F - E + V = 2$	Simple Solids
$F - E + V - L = B - G$	Open Objects

F=Face	E=Edge	V=Vertices
B=Bodies	L=Faces' inner Loop	G=Genus

Euler Operations

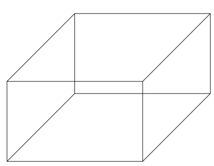
- A connected structure of vertices, edges and faces that always satisfies Euler's formula is known as **Euler object**.
- The process that adds and deletes these boundary components is called an **Euler operation**

Applicability of Euler formula to solid objects:

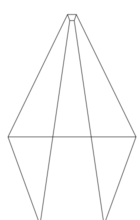
- At least three edges must meet at each vertex.
- Each edge must share two and only two faces
- All faces must be simply connected (homomorphic to disk) with no holes and bounded by single ring of edges.
- The solid must be simply connected with no through holes

Validity Checking for Simple Solids

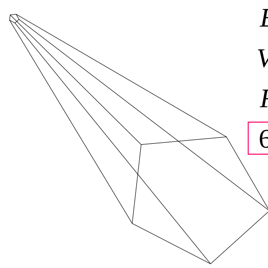
$$F - E + V = 2 \quad \text{Simple Solids}$$



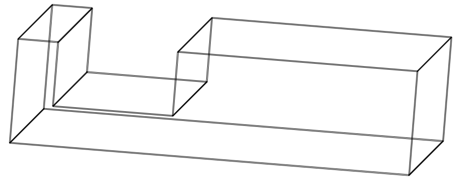
$E = 12$
$V = 8$
$F = 6$
$6 - 12 + 8 = 2$



$E = 8$
$V = 5$
$F = 5$
$5 - 8 + 5 = 2$



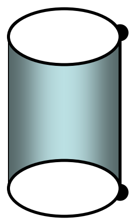
$E = 10$
$V = 6$
$F = 6$
$6 - 10 + 6 = 2$

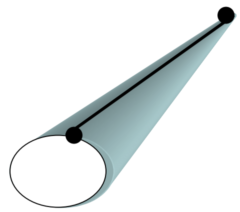


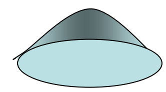
$E = 24$
$V = 16$
$F = 10$
$10 - 24 + 16 = 2$

Validity Checking for Simple Solids

$F - E + V = 2$ Simple Solids







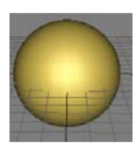
$E = 3$
$V = 2$
$F = 3$
$3 - 3 + 2 = 2$

$E = 2$
$V = 2$
$F = 2$
$2 - 2 + 2 = 2$

$E = 2$
$V = 2$
$F = 2$
$2 - 2 + 2 = 2$

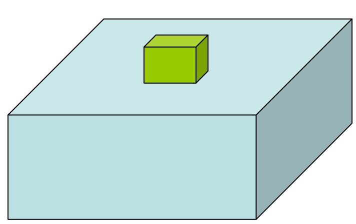
Loops (rings),Genus & Bodies

- **Genus zero**



- **Genus one**

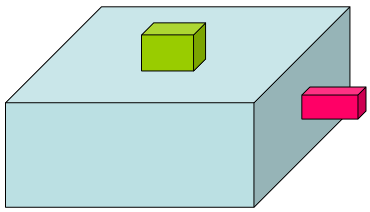
- **Genus two**



- **One inner loop**

Validity Checking for Polyhedra with inner loops

$$F - E + V - L = 2(B - G)$$ General



$E = 36$

$F = 16$

$V = 24$

$L = 2$

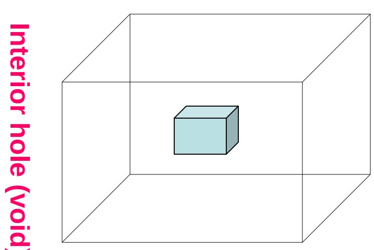
$B = 1$

$G = 0$

$$16 - 36 + 24 - 2 = 2(1 - 0) = 2$$

Validity Checking for Polyhedra with holes

$$F - E + V - L = 2(B - G)$$ General



$E = 24$

$F = 12$

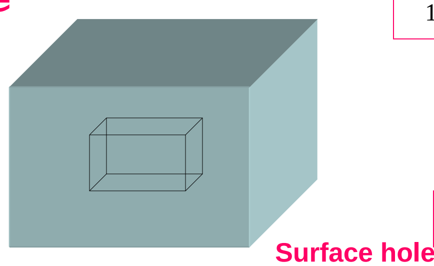
$V = 16$

$L = 0$

$B = 2$

$G = 0$

$$12 - 24 + 16 - 0 = 2(2 - 0) = 4$$



$E = 24$ $F = 11$

$V = 16$ $L = 1$

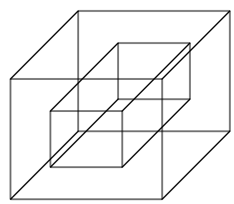
$B = 1$ $G = 0$

$$11 - 24 + 16 - 1 = 2(1 - 0) = 2$$

Validity Checking for Polyhedra with through holes (handles)

$$F - E + V - L = 2(B - G) \qquad \text{General}$$

Through hole



$E = 24$

$F = 10$

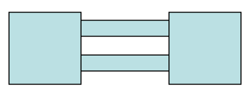
$V = 16$

$L = 2$

$B = 1$

$G = 1$

$10 - 24 + 16 - 2 = 2(1-1) = 0$



Handles/through hole

$E = 48 \quad F = 20$

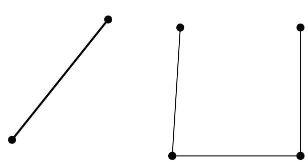
$V = 32 \quad L = 4$

$B = 1 \quad G = 1$

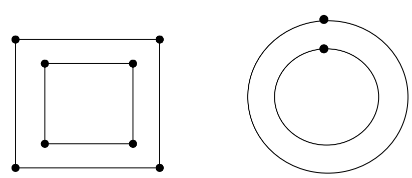
$20 - 48 + 32 - 4 = 2(1-1) = 0$

Validity Checking for Open Objects

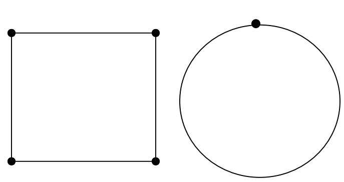
$$F - E + V - L = B - G$$



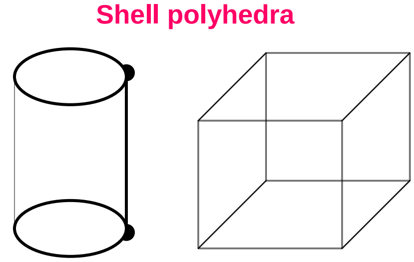
Wireframe polyhedra



Shell polyhedra



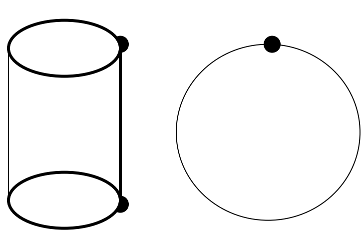
Lamina polyhedra



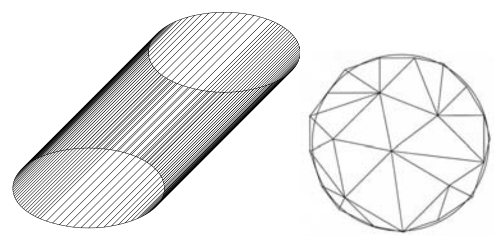
Open three dimensional polyhedra

Exact Vs Faceted B-rep Schemes

- **Exact B-rep**: If the curved objects are represented by way of equations of the underlying curves and sufraces, then the scheme is **Exact B-rep**.
- **Approximate or faceted B-rep** :In this scheme of boundary representation any curved face divided into planar faces. It is also know as **tessellation** representation.



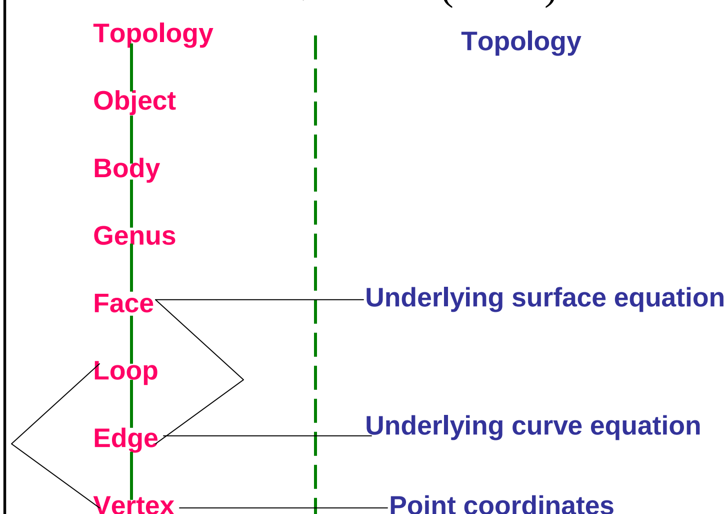
Exact B-rep: Cylinder and Sphere



Faceted cylinder and sphere

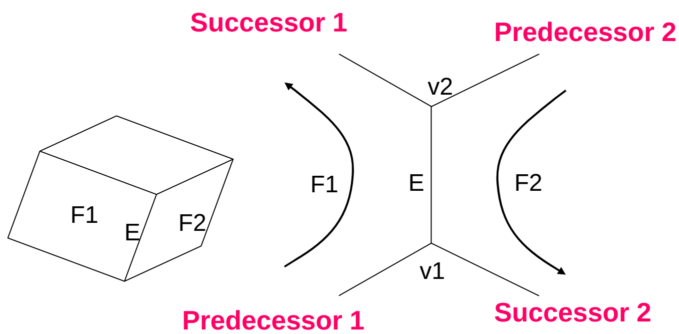
Data structure for B-rep models

$$F - E + V - L = 2(B - G) \qquad \text{General}$$



Winged Edge Data structure

All the adjacency relations of each edge are described explicitly. An edge is adjacent to exactly two faces and hence it is component in two loops, one for each face.



As each face is orientable, edges of the loops are traversed in a given direction. The wingded edge data structure is efficient in object modifications (addition, deletion of edges, Euler operations).

Building Operations

$$F - E + V - L = 2(B - G) \qquad \text{General}$$

The basis of the Euler operations is the above equation. M and K stand for Make and Kill respectively.

Operation	Operator	Complement	Description
Initiate Database and begin creation	MBFV	KBFV	Make Body Face Vertex
Create edges and vertices Create edges and faces	MEV MEKL MEF MEKBFL MFKLG	KEV KEML KEF KEMBFL KFMLG	Make Edge Vertex Make Edge Kill Loop Make Edge Face Make Edge Kill Body, Face Loop Make Edge Kill Loop Genus
Glue	KFEVMG KFEVB	MFEVKG MFEVB	Kill Face Edge Vertex Make Genus Kill Face Edge Vertex Body
Composite Operations	MME ESPLIT KVE	KME ESQUEEZE	Make Multiple Edges Edge Split Kill Vertex Edge

Transition States of Euler Operations

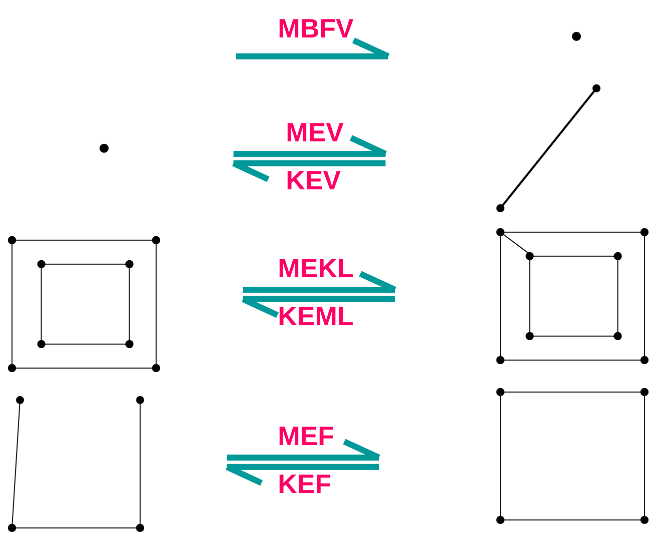
$$F - E + V - L = 2(B - G) \qquad \text{General}$$

While creating B-rep models at each stage we use Euler operators and ensure the validity.

Operator	F	E	V	L	B	G
MBFV	1	0	1	0	1	0
MEV	0	1	1	0	0	0
MEKL	0	1	0	-1	0	0
MEF	1	1	0	0	0	0
MEKBFL	-1	1	0	-1	-1	0
MFKLG	1	0	0	-1	0	-1
KFEVMG	-2	-n	-n	0	0	1
KFEVB	-2	-n	-n	0	-1	0
MME	0	n	n	0	0	9
ESPLIT	0	1	1	0	0	9
KVE	-(n-1)	-n	-1	0	0	9

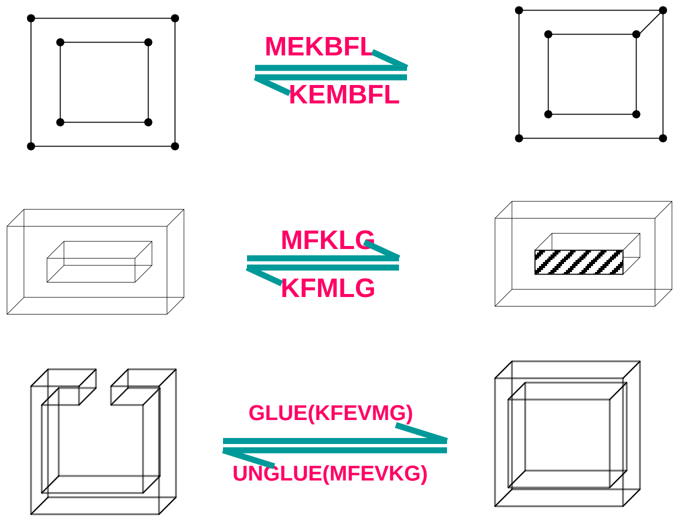
Euler Operations

$$F - E + V - L = 2(B - G)$$



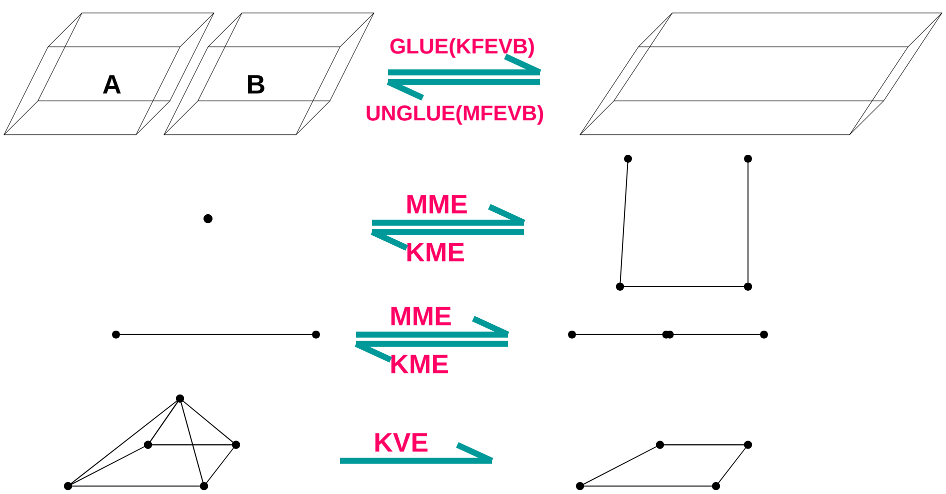
Euler Operations

$$F - E + V - L = 2(B - G)$$

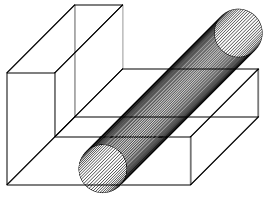


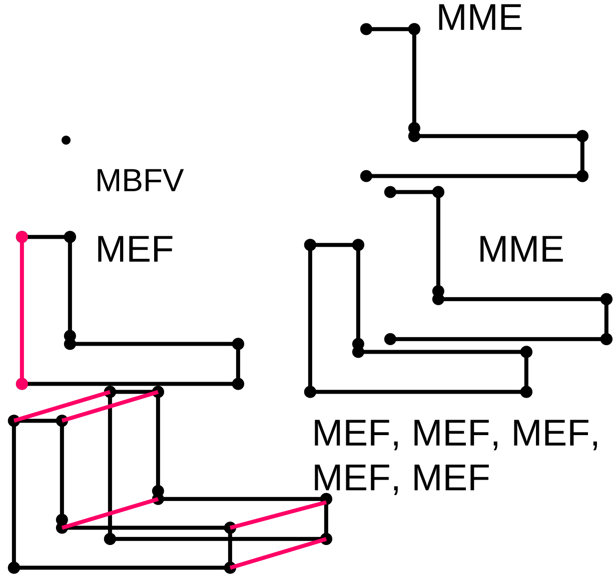
Euler Operations

$$F - E + V - L = 2(B - G)$$



Building operations





Merits and Demerits of Euler Operations

If the operator acts on a valid topology and the state transition it generates is valid, then the resulting topology is a valid solid. Therefore, Euler's law is never verified explicitly by the modeling system.

- **Merits**:
  - They ensure creating valid topology
  - They provide full generality and reasonable simplicity
  - They achieve a higher semantic level than that of manipulating faces, edges and vertices directly
- **Demerits** :
  - They do not provide any geometrical information to define a solid polyhedron
  - They do not impose any restriction on surface orientation, face planarity, or surface self intersection

Advantages and Disadvantages of B-rep

Advantages:

- It is historically a popular modeling scheme related closely to traditional drafting
- It is very appropriate tool to construct quite unusual shapes like aircraft fuselage and automobile bodies that are difficult to build using primitives
- It is relatively simple to convert a B-rep model into a wireframe model because its boundary definition is similar to the wireframe definitions
- In applications B-rep algorithms are reliable and competitive to CSG based algorithms

Disadvantages :

- It requires large storage space as it stores the explicit definitions of the model boundaries
- It is more verbose than CSG
- Faceted B-rep is not suitable for manufacturing applications