3D TRANSFORMATIONS

1. Linear 3D Transformations:
   Translation, Rotation, Scaling
   Shearing, Reflection
2. Perspective Transformations

Transformations in 3 dimensions

- Geometric transformations are mappings from one coordinate system onto itself.
- The geometric model undergoes change relative to its MCS (Model Coordinate System)
- The Transformations are applied to an object represented by point sets.
- Rigid Body Motion: The relative distances between object particles remain constant
- Affine and Non-Affine maps
- Transformed point set \( X^* = f(P, \text{ transformation parameters}) \)
Homogeneous coordinates in 3 dimensions
- A point in homogeneous coordinates \((x, y, z, h), h \neq 0\), corresponds to the 3-D vertex \((x/h, y/h, z/h)\) in Cartesian coordinates.
- Homogeneous coordinates in 3D give rise to 4 dimensional position vector.

Generalized 4 x 4 transformation matrix in homogeneous coordinates

- Perspective transformations
- Linear transformations – local scaling, shear, rotation / reflection
- Translations \(l, m, n\) along x, y, and z axis
- Overall scaling
3D Scaling

\[ [T_s] = \begin{bmatrix}
    a & 0 & 0 & 0 \\
    0 & e & 0 & 0 \\
    0 & 0 & j & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} \]

Ex: Required scaling to scale the RPP to a unit cube is \( \frac{1}{2} \), \( \frac{1}{3} \), 1

\[ [X'] = [T_s] [X] = \begin{bmatrix}
    0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 \\
    0 & 0 & 3 & 3 & 0 & 0 & 3 & 3 \\
    1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
    1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix} \]

Overall Scaling

\[ [T_s] = \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & s
\end{bmatrix} \]

\[ [T_s][X] = \begin{bmatrix}
    x' & y' & z' & s
\end{bmatrix}^T = \begin{bmatrix}
    x'/s & y'/s & z'/s & 1
\end{bmatrix}^T \]

Ex: Uniformly scale the unit cube by a factor of 2 requires \( s = \frac{1}{2} \)

\[ [X'] = \begin{bmatrix}
    0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 \\
    0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\
    2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
    1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix} \]
3D SHEARING

\[
[T_{SH}] = \begin{bmatrix}
d & g & 0 \\
b & 1 & i \\
c & f & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
[T_s][X] = [x + yd + gz, bx + y + iz, cx + fy + z, 1]^T
\]

Ex: Uniformly scale the unit cube by a factor of 2 requires \(d = -0.75, g=0.5, i=1, b=-0.85, c=0.25, f=0.7\)

\[
[X] = \begin{bmatrix}
0.5 & 1.5 & 0.75 & -0.25 & 0 & 1 & 0.25 & -0.75 \\
1 & 0.15 & 1.15 & 2 & 0 & -0.85 & 0.15 & 1 \\
1 & 1.25 & 1.95 & 1.7 & 0 & 0.25 & 0.95 & 0.7 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

3D ROTATIONS – (i) Rotation about z-axis

We are already familiar with rotation about z-axis (in 2D rotations)

\[
[T_{Rz}] = \begin{bmatrix}
\cos \psi & \sin \psi & 0 & 0 \\
-\sin \psi & \cos \psi & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\text{Det}[T] = +1
\]

The position vector is assumed to be a row vector in right handed system
3D ROTATIONS – (ii) Rotation about x-axis
Similarly we can obtain rotation matrix about x-axis

\[
[T_{Rx}] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta & 0 \\
0 & -\sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

3D ROTATIONS – (iii) Rotation about y-axis
We can obtain rotation about y-axis as

\[
[T_{Ry}] = \begin{bmatrix}
\cos \phi & 0 & -\sin \phi & 0 \\
0 & 1 & 0 & 0 \\
\sin \phi & 0 & \cos \phi & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Example – for the RPP given by the matrix below obtain \(-90\) rotation about \(x\) -axis

\[
[X] = \begin{bmatrix}
0 & 3 & 3 & 0 & 0 & 3 & 3 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

\[
[T_{Rx}] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
[X'] = \begin{bmatrix}
0 & 3 & 3 & 0 & 0 & 3 & 3 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -2 & 0 & 0 & -2 & -2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

Example – for the RPP given by the matrix below obtain \(90\) rotation about \(y\) -axis

\[
[X] = \begin{bmatrix}
0 & 3 & 3 & 0 & 0 & 3 & 3 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

\[
[T_{Ry}] = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
[X'] = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\
0 & -3 & -3 & 0 & 0 & -3 & -3 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]
3D REFLECTIONS – As in 2D, we can perform 3D transformations about a plane now.

Rotation of 180° about an axis passing through origin out into 4-D space and projection back onto 3D space.

Through x-y plane

Similarly through y-z and x-z planes are

Example – for the RPP given by the matrix below obtain 3D reflection through xy - plane
COMBINATION OF TRANSFORMATIONS – As in 2D, we can perform a sequence of 3D linear transformations.

This is achieved by concatenation of transformation matrices to obtain a combined transformation matrix.

A combined matrix \( [T] \) is achieved by concatenation of transformation matrices to obtain a combined transformation matrix

Where \([T_i]\) are any combination of

- Translation
- Scaling
- Shearing (linear trans. but not perspective)
- Rotation (perspective transformation)
- Reflection (Results in loss of info)

Example – Transform the given position vector \([3 2 1 1]\) by the following sequence of operations

(i) Translate by \(-1\), \(-1\), \(-1\) in x, y, and z respectively

(ii) Rotate by \(+30^\circ\) about x-axis and \(+45^\circ\) about y axis

The concatenated transformation matrix is:

\[
[T] = [T_{tr}] [T_{rx(30)}] [T_{ry(45)}] =
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\cos \theta & 0 & \sin \theta & 0 \\
0 & 1 & 0 & 0 \\
-\sin \theta & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
[T][X] = \begin{bmatrix}
0.707 & 0.354 & 0.612 & -1.673 \\
0 & 0.866 & -0.5 & -0.366 \\
-0.707 & 0.354 & 0.612 & -0.259 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
3 \\
2 \\
1 \\
1 \\
\end{bmatrix}
= \begin{bmatrix}
1.768 \\
0.866 \\
-1.061 \\
1 \\
\end{bmatrix}
\]

Rotation about an axis parallel to a coordinate axis

- Translate the axis(line) to coincide with the axis to which it is parallel
- Rotate the object by required angle
- Translate the object back to its original position

**Example** – Consider the following cube. Rotate it by 30° about an axis \( x' \) passing through its centroid \( l=3/2, m=3/2 \) and \( n=3/2 \)

\[
[T] = \begin{bmatrix}
1 & 0 & 0 & -l \\
0 & 1 & 0 & -m \\
0 & 0 & 1 & -n \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & l \\
0 & 1 & 0 & m \\
0 & 0 & 1 & n \\
\end{bmatrix} \begin{bmatrix}
l & 1 & 1 & 1 & 2 & 2 & 1 \\
m & 1 & 1 & 2 & 2 & 1 & 2 \\
n & 2 & 2 & 2 & 2 & 1 & 1 \\
\end{bmatrix}
\]

\( l=3/2, m=3/2 \) and \( n=3/2 \)
ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

Make the arbitrary axis coincide with one of the coordinate axes.

- Consider an arbitrary axis passing through a point \((x_0, y_0, z_0)\)

**Procedure**

- Translate \((x_0, y_0, z_0)\) so that the point is at origin
- Make appropriate rotations to make the line coincide with one of the axes, say z-axis
- Rotate the object about z-axis by required angle
- Apply the inverse of step 2
- Apply the inverse of step 1
- Coinciding the arbitrary axis with any axis the rotations are needed about other two axes

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**ROTATION ABOUT AN ARBITRARY AXIS IN SPACE**

- To calculate the angles of rotations about the x and y axes consider direction cosines \((c_x, c_y, c_z)\)

\[
d = \sqrt{c_x^2 + c_z^2} \quad \cos \alpha = \frac{c_x}{d} \quad \sin \alpha = \frac{c_y}{d}
\]

\[
\cos \beta = d \quad \sin \beta = c_x
\]
The complete sequence of operations can be summarised as follows:

\[
[T] = [Tr][R_\alpha][R_\beta][R][R_\beta]^{-1}[R_\alpha]^{-1}[Tr]^{-1}
\]

\[
[Tr] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-x_0 & -y_0 & -z_0 & 1
\end{bmatrix}
\]

\[
[R_\alpha] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha & 0 \\
0 & \sin \alpha & \cos \alpha & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
[R_\beta] = \begin{bmatrix}
\cos(-\beta) & 0 & -\sin(-\beta) & 0 \\
0 & 1 & 0 & 0 \\
\sin(-\beta) & 0 & \cos(-\beta) & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Substituting for the respective angles we obtain the following matrices:

\[
[T] = [Tr][R_\alpha][R_\beta][R][R_\beta]^{-1}[R_\alpha]^{-1}[Tr]^{-1}
\]

\[
[R_\alpha] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha & 0 \\
0 & \sin \alpha & \cos \alpha & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

- Direction cosines of an arbitrary axis when two points on the line are known:

\[
\begin{pmatrix}
(x_1, y_1, z_1) \\
(x_0, y_0, z_0)
\end{pmatrix}
\]

\[
\mathbf{V} = \begin{pmatrix}
(x_1-x_0) & (y_1-y_0) & (z_1-z_0)
\end{pmatrix}
\]

\[
\begin{bmatrix}
c_x & c_y & c_z
\end{bmatrix} = \begin{bmatrix}
\frac{(x_1-x_0)}{\sqrt{(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2}} & \frac{(y_1-y_0)}{\sqrt{(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2}} & \frac{(z_1-z_0)}{\sqrt{(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2}}
\end{bmatrix}
\]

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

- Find the transformation matrix for reflection with respect to the plane passing through the origin and having normal vector \( \mathbf{n} = i+j+k \)

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\mathbf{n} = \begin{pmatrix} 1 \ 1 \ 1 \end{pmatrix}, \quad |\mathbf{n}| = \sqrt{3}
\]

\[
\begin{bmatrix}
c_x & c_y & c_z
\end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \ 1/\sqrt{3} \ 1/\sqrt{3} \end{bmatrix}
\]