AML710 CAD LECTURE 6

3D TRANSFORMATIONS

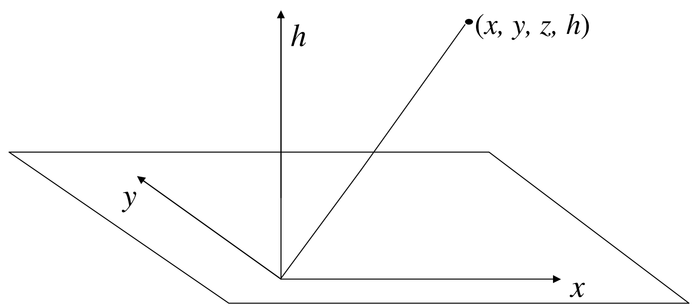
1. Linear 3D Transformations: Translation, Rotation, Scaling Shearing, Reflection
2. Perspective Transformations

Transformations in 3 dimensions

- Geometric transformations are mappings from one coordinate system onto itself.
- The geometric model undergoes change relative to its MCS (Model Coordinate System)
- The Transformations are applied to an object represented by point sets.
- Rigid Body Motion: The relative distances between object particles remain constant
- Affine and Non-Affine maps
- Transformed point set *X* = f(P, transformation parameters)*

Homogeneous coordinates in 3 dimensions

- A point in homogeneous coordinates (x, y, z, h), $h \neq 0$, corresponds to the 3-D vertex ($x/h, y/h, z/h$) in Cartesian coordinates.
- Homogeneous coordinates in 3D give rise to 4 dimensional position vector.



- Generalized 4 x 4 transformation matrix in homogeneous coordinates

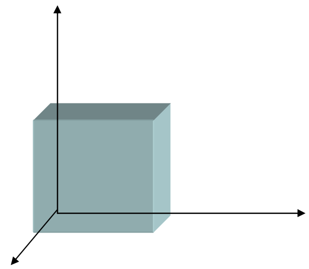
$$[T] = \begin{bmatrix} a & d & g & p \\ b & e & i & q \\ c & f & j & r \\ l & m & n & s \end{bmatrix}$$

Local Scaling

- Perspective transformations
- Linear transformations – local scaling, shear, rotation / reflection
- Translations l, m, n along x, y, and z axis
- Overall scaling

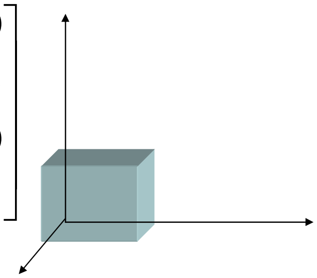
➢ 3D Scaling

$$[T_s] = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



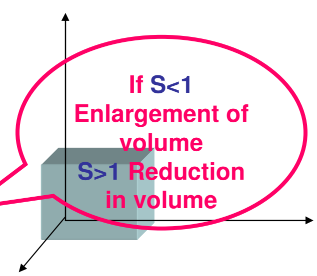
Ex: Required scaling to scale the RPP to a unit cube is ½, 1/3, 1

$$[X'] = [T_s]\begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 & 0 & 3 & 3 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



➢ Overall Scaling

$$[T_s] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix}$$

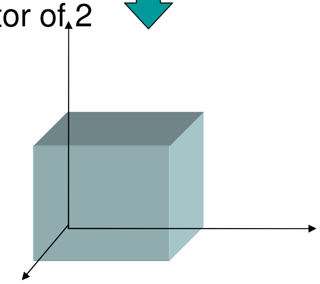


$$[T_s][X] = [x' \quad y' \quad z' \quad s]^T$$

$$= [x'/s \quad y'/s \quad z'/s \quad 1]^T$$

Ex: Uniformly scale the unit cube by a factor of 2 requires s= ½

$$[X'] = \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



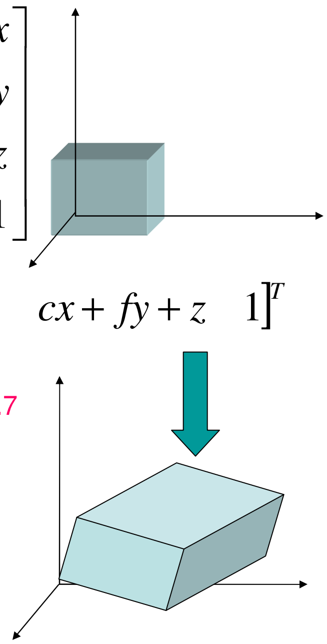
- 3D SHEARING

$$[T_{SH}] = \begin{bmatrix} 1 & d & g & 0 \\ b & 1 & i & 0 \\ c & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$[T_s][X] = \begin{bmatrix} x + yd + gz & bx + y + iz & cx + fy + z & 1 \end{bmatrix}^T$$

Ex: Uniformly scale the unit cube by a factor of 2 requires d= -0.75, g=0.5,i=1, b=-0.85,c=0.25,f=0.7

$$[X'] = \begin{bmatrix} 0.5 & 1.5 & 0.75 & -0.25 & 0 & 1 & 0.25 & -0.75 \\ 1 & 0.15 & 1.15 & 2 & 0 & -0.85 & 0.15 & 1 \\ 1 & 1.25 & 1.95 & 1.7 & 0 & 0.25 & 0.95 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

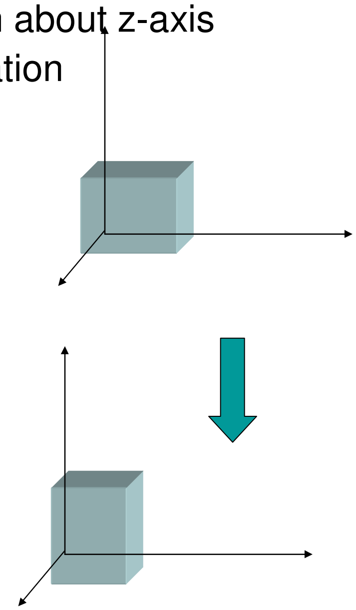


- 3D ROTATIONS – (i) Rotation about z-axis

We are already familiar with rotation about z-axis (in 2D rotations)

$$[T_{Rz}] = \begin{bmatrix} \cos\psi & \sin\psi & 0 & 0 \\ -\sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Det[T]=+1

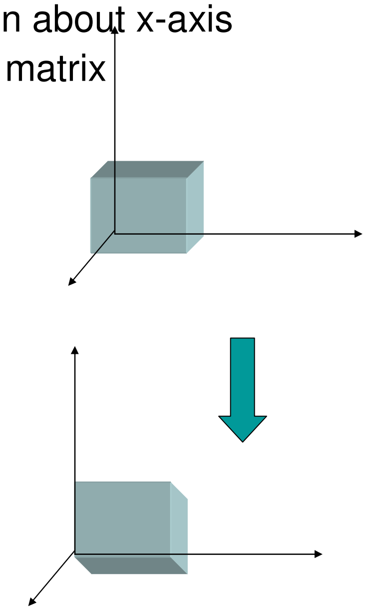


The position vector is assumed to be a row vector in right handed system

3D ROTATIONS – (ii) Rotation about x-axis

Similarly we can obtain rotation matrix about x-axis

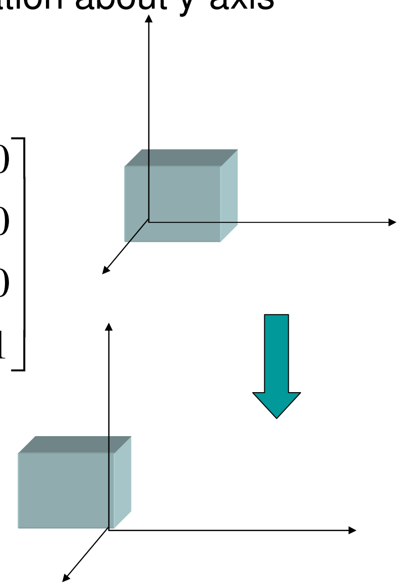
$$[T_{Rx}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta & 0 \\ 0 & -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



3D ROTATIONS – (iii) Rotation about y-axis

We can obtain rotation about y-axis as

$$[T_{Ry}] = \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

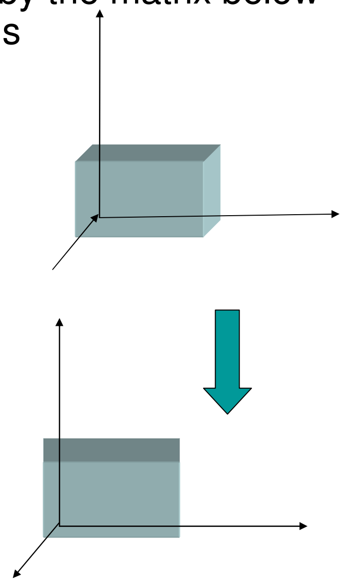


- Example – for the RPP given by the matrix below obtain -90 rotation about x -axis

$$[X]=\begin{bmatrix} 0 & 3 & 3 & 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[T_{Rx}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[X'] = \begin{bmatrix} 0 & 3 & 3 & 0 & 0 & 3 & 3 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 & -2 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

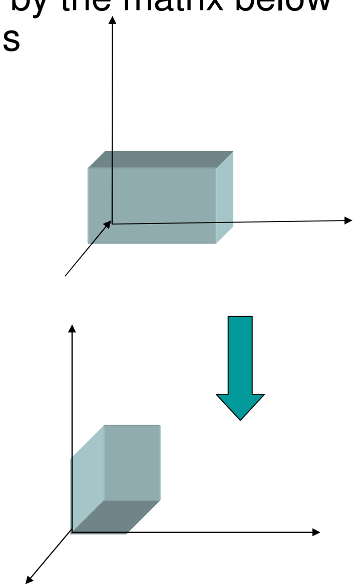


- Example – for the RPP given by the matrix below obtain 90 rotation about y -axis

$$[X]=\begin{bmatrix} 0 & 3 & 3 & 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[T_{Ry}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[X'] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\ 0 & -3 & -3 & 0 & 0 & -3 & -3 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



- 3D REFLECTIONS – As in 2D, we can perform 3D transformations about a plane now.
- Rotation of 180° about an axis passing through origin out into 4-D space and projection back onto 3D space.

Det[T]=-1

Through x-y plane

$$[T_{xy}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly through y-z and x-z planes are

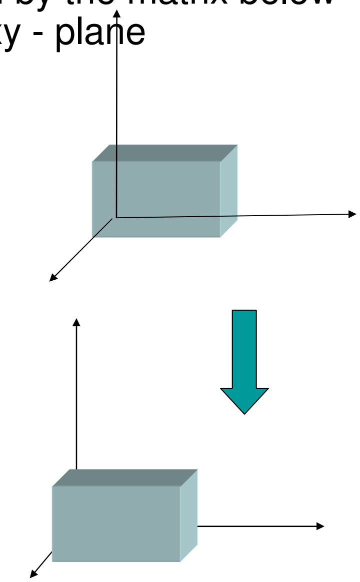
$$[T_{yz}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [T_{xz}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Example – for the RPP given by the matrix below obtain 3D reflection through xy - plane

$$[X] = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & -2 & -2 & -2 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[T_{xy}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[X] = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



- COMBINATION OF TRANSFORMATIONS – As in 2D, we can perform a sequence of 3D linear transformations.
- This is achieved by concatenation of transformation matrices to obtain a combined transformation matrix

A combined matrix $[T][X] = [X][T_1][T_2][T_3][T_4]\ldots[T_n]$

Where $[T_i]$ are any combination of

- Translation
- Scaling
- Shearing
- Rotation
- Reflection

(Scaling, Shearing, Rotation, Reflection) } linear trans. but not perspective transformation (Results in loss of info)

- Example – Transform the given position vector [3 2 1 1] by the following sequence of operations
(i) Translate by –1, -1, -1 in x, y, and z respectively
(ii) Rotate by +30° about x-axis and +45° about y axis
The concatenated transformation matrix is:

$$[T] = [T_{tr}][T_{rx(30)}][T_{ry(45)}] =$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T][X] = \begin{bmatrix} 0.707 & 0.354 & 0.612 & -1.673 \\ 0 & 0.866 & -0.5 & -0.366 \\ -0.707 & 0.354 & 0.612 & -0.259 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.768 \\ 0.866 \\ -1.061 \\ 1 \end{bmatrix}$$

Rotation about an axis parallel to a coordinate axis

- Translate the axis(line) to coincide with the axis to which it is parallel
- Rotate the object by required angle
- Translate the object back to its original position

➢ Example – Consider the following cube. Rotate it by 30º about an axis x' passing through its centroid

$$[T] = \begin{bmatrix} 1 & 0 & 0 & -l \\ 0 & 1 & 0 & -m \\ 0 & 0 & 1 & -n \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

l=3/2,m3/2 and n=3/2

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 0.951 \\ 0 & 0.5 & 0.866 & -0.549 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 \\ 0.817 & 0.817 & 1.683 & 1.683 & 1.317 & 1.317 & 2.183 & 2.183 \\ 1.683 & 1.683 & 2.183 & 2.183 & 0.817 & 0.817 & 1.317 & 1.317 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

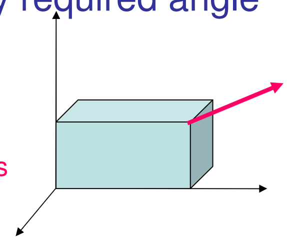
ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

Make the arbitrary axis coincide with one of the coordinate axes.

- Consider an arbitrary axis passing through a point (x_0, y_0, z_0)

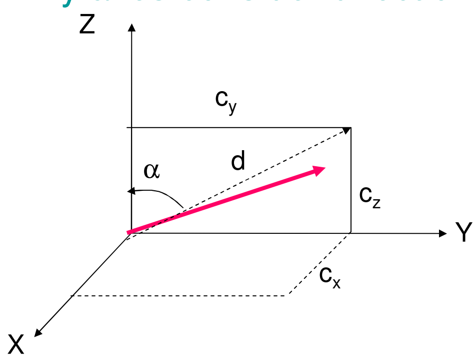
Procedure

- Translate (x_0, y_0, z_0) so that the point is at origin
- Make appropriate rotations to make the line coincide with one of the axes, say z-axis
- Rotate the object about z-axis by required angle
- Apply the inverse of step 2
- Apply the inverse of step 1
- Coinciding the arbitrary axis with any axis the rotations are needed about other two axes

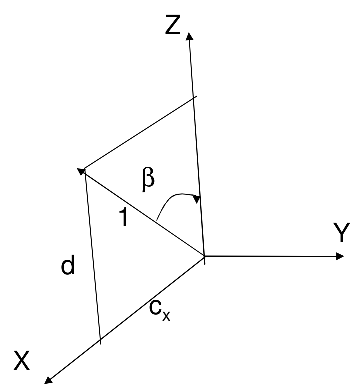


ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

- To calculate the angles of rotations about the x and y axes consider direction cosines (c_x, c_y, c_z)



$d = \sqrt{c_y^2 + c_z^2}$ $\cos\alpha = \frac{c_z}{d}$ $\sin\alpha = \frac{c_y}{d}$



$\cos\beta = d$ $\sin\beta = c_x$

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

- The complete sequence of operations can be summarised as follows:

$$[T]=[Tr][R_\alpha][R_\beta][R][R_\beta]^{-1}[R_\alpha]^{-1}[Tr]^{-1}$$

$$[Tr]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-x_0&-y_0&-z_0&1\end{bmatrix}\quad [R_\alpha]=\begin{bmatrix}1&0&0&0\\0&\cos\alpha&\sin\alpha&0\\0&-\sin\alpha&\cos\alpha&0\\0&0&0&1\end{bmatrix}\quad [R_\beta]=\begin{bmatrix}\cos(-\beta)&0&-\sin(-\beta)&0\\0&1&0&0\\\sin(-\beta)&0&\cos(-\beta)&0\\0&0&0&1\end{bmatrix}$$

$$[Tr]^{-1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\x_0&y_0&z_0&1\end{bmatrix}\quad [R_\alpha]^{-1}=\begin{bmatrix}1&0&0&0\\0&\cos(-\alpha)&\sin(-\alpha)&0\\0&-\sin(-\alpha)&\cos(-\alpha)&0\\0&0&0&1\end{bmatrix}\quad [R_\beta]^{-1}=\begin{bmatrix}\cos\beta&0&-\sin\beta&0\\0&1&0&0\\\sin\beta&0&\cos\beta&0\\0&0&0&1\end{bmatrix}$$

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

- Substituting for the respective angles we obtain the following matrices:

$$[T]=[Tr][R_\alpha][R_\beta][R][R_\beta]^{-1}[R_\alpha]^{-1}[Tr]^{-1}$$

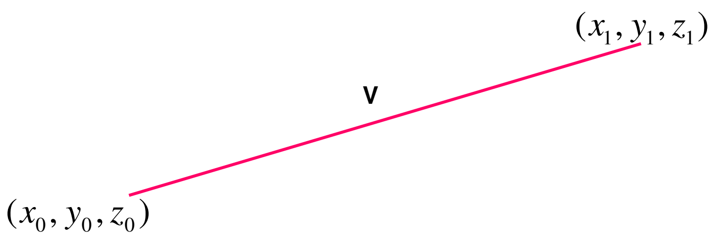
$$[R_\alpha]=\begin{bmatrix}1&0&0&0\\0&\cos\alpha&\sin\alpha&0\\0&-\sin\alpha&\cos\alpha&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&c_z/d&c_y/d&0\\0&-c_y/d&c_z/d&0\\0&0&0&1\end{bmatrix}$$

$$[R_\beta]=\begin{bmatrix}\cos(-\beta)&0&-\sin(-\beta)&0\\0&1&0&0\\\sin(-\beta)&0&\cos(-\beta)&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}d&0&c_x&0\\0&1&0&0\\-c_x&0&d&0\\0&0&0&1\end{bmatrix}$$

$$[R]=\begin{bmatrix}\cos\delta&\sin\delta&0&0\\-\sin\delta&\cos\delta&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

- Direction cosines of a an arbitrary axis when two points on the line are known:

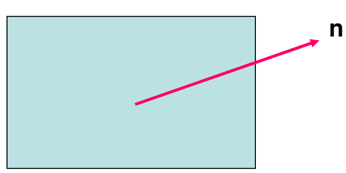


$$[V] = \left[(x_1 - x_0) \quad (y_1 - y_0) \quad (z_1 - z_0)\right]$$

$$[c_x \quad c_y \quad c_z] = \left[\frac{(x_1 - x_0) \quad (y_1 - y_0) \quad (z_1 - z_0)}{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}}\right]$$

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

- Find the transformation matrix for reflection with respect to the plane passing through the origin and having normal vector **n** = i+j+k



$$|\, n \,| = \left[1^2 \quad 1^2 \quad 1^2\right]^{1/2} = \sqrt{3}$$

$$[c_x \quad c_y \quad c_z] = \left[\frac{1 \quad 1 \quad 1}{\sqrt{3}}\right]$$