Thermal postbuckling analysis of FGM skew plates

T. Prakash\textsuperscript{a}, M.K. Singha\textsuperscript{a,∗}, M. Ganapathi\textsuperscript{b}

\textsuperscript{a} Department of Applied Mechanics, Indian Institute of Technology Delhi, New Delhi - 110 016, India
\textsuperscript{b} Institute of Armament Technology, Girinagar, Pune - 411 025, India

Received 29 March 2006; received in revised form 26 January 2007; accepted 21 February 2007
Available online 6 April 2007

Abstract

In this paper, the postbuckling behavior of functionally graded material (FGM) skew plates under thermal load is investigated based on the shear deformable finite element approach. The material is graded in the thickness direction according to a power-law distribution in terms of the volume fractions of the constituents. The Mori–Tanaka homogenization method is used to estimate the effective material properties from the volume fractions and the properties of the constituent materials. The temperature field is assumed to be uniform over the plate surface and varies in the thickness direction only. The nonlinear governing equations derived based on von Karman’s assumptions are solved employing the direct iterative technique. The existence of bifurcation-type of buckling of FGM plates is examined by considering different parameters such as the constituent gradient index, temperature distribution, thickness-to-span ratio, aspect ratio, skew angle, and boundary conditions. The effect of temperature dependent material properties on the thermal postbuckling characteristics of FGM skew plates is also studied.

© 2007 Elsevier Ltd. All rights reserved.

Keywords: FGM plate; Volume fraction index; Thermal load; Post-buckling; Extension–bending coupling; Temperature dependent material properties; Finite element

1. Introduction

Thin-walled structural members of aerospace, defence, nuclear reactors, and other high performance application areas that are subjected to thermal load in addition to mechanical in-plane load, may undergo thermo-elastic instability. Hence, the nonlinear behavior of such members, like plates and shells, has to be understood for their optimum design. Consequently, the thermo-mechanical stability characteristics of isotropic and composite plates have received a considerable amount of attention from researchers in last two decades. A parallel research effort on high performance material resulted in functionally graded materials (FGM) [1,2] that produce continuously changing thermal and mechanical properties at the macroscopic level and offer many advantages over the traditional methods of tailoring the material properties. The advantages of using these materials in thin-walled structural members are that they are able to withstand a high-temperature gradient environment while maintaining their structural integrity, and that they avoid the interface problem that exists in homogeneous composites. In view of this, the study of thermal stability behavior of FGM plates has received considerable attention in the literature recently.

Birman [3] has studied the buckling of FGM plates subjected to uniaxial compression. Javaheri and Eslami [4,5] have carried out mechanical and thermal buckling analyses of FGM plates using Kirchhoff’s thin plate theory, whereas Wu [6] has examined the effect of shear deformation on the thermal buckling of FGM plates. Chen and Liew [7] have examined the buckling of rectangular FGM plates subjected to in-plane edge loads. Najafizadeh and Eslami [8] have presented the thermal stability of circular functionally graded plates. It is observed that most of the abovementioned work has dealt with the evaluation of the critical mechanical/thermal buckling loads of rectangular/circular FGM plates using the small deflection theory.

FGM plates are, in general, non-symmetric-through-thickness, as the material properties vary through thickness. Furthermore, the variation of the thermal load in such FGM plates is practically nonlinear through thickness. The bifurcation type of instability is examined for non-symmetric-through-thickness panel problems by Leissa [9]. Qatu and

---

∗ Corresponding author. Tel.: +91 1126596445; fax: +91 1126581119. E-mail address: maloy@am.iitd.ernet.in (M.K. Singha).

0141-0296/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.engstruct.2007.02.012
Leissa [10] using unsymmetric cross-ply laminated plates subjected to an in-plane load. Singha et al. [11] have made similar studies while dealing with composite plates under a non-symmetric thermal load. It is observed that, irrespective of symmetric/non-symmetric loads and plate geometry in the thickness direction, the clamped plates exhibit the bifurcation type of buckling and can be studied based on eigenvalue analysis. For the non-symmetric situation with boundary conditions other than clamped one, nonlinear analyses have to be introduced because of the bending behavior of the plate due to extension–bending coupling, irrespective of the magnitudes of the applied loads. However, such problems for FGM plates under mechanical/thermal loads have been investigated employing eigen-buckling analyses [3–8], which may not reveal the actual behavior. Most recently, Liew et al. [12] studied the thermal postbuckling characteristics of laminated plates comprising functionally graded materials using the differential quadrature method, whereas Liew et al. [13] and Shen [14] have discussed the existence of a bifurcation type of instability for piezoelectric FGM plates under thermo-electro-mechanical loading. The thermal postbuckling path for shell structures has also been investigated [29–31] in the literature. The postbuckling behavior of plates using the higher order shear deformation theory [32] and the 3D finite element method [33] have been attempted. It may be inferred from available works that a comprehensive insight into the thermal postbuckling behavior of FGM plate structures based on an appropriate model seems to be missing in the literature. In view of this, a nonlinear analysis including extension–bending coupling is necessary to understand the actual characteristics of FGM plates under a thermal load. Furthermore, to the best of the authors’ knowledge, the work on the thermal postbuckling behaviors of FGM skew plates which find wide applications in the aerospace industry is not yet commonly available in the literature.

Here, an eight-noded \( C_0 \) shear flexible quadrilateral plate element developed based on the consistency approach [15, 16] is used to analyze the nonlinear behavior of FGM skew plates subjected to thermal loads. The material properties are graded in the thickness direction according to the power-law distribution in terms of the volume fractions of the constituents of the material. The effective material properties are estimated using the Mori–Tanaka homogenization method. The nonlinear governing equations based on von Karman’s assumptions are solved with the Newton–Raphson method to study the nonlinear behavior of FGM plates under a thermal load. The influences of the material gradient index, thickness of the plate, through-thickness-temperature distribution (constant, linear, and nonlinear), boundary condition, skew-angle, and aspect ratio on the nonlinear characteristics of functionally graded skew plates are highlighted.

2. Formulation

A functionally graded material skew plate (length \( a \), width \( b \), and thickness \( h \)) made by mixing two distinct material phases, for example a metal and a ceramic, is considered with the coordinates \( x, y \) along the in-plane directions and \( z \) along the thickness direction. The material on the top surface \((z = h/2)\) of the plate and in the bottom surface \((z = −h/2)\) of the plate is ceramic and metal, respectively. The locally effective material properties are evaluated using a homogenization method that is based on the Mori–Tanaka scheme [17,18]. The effective bulk modulus \( B \) and shear modulus \( G \) of the functionally gradient material evaluated using the Mori–Tanaka estimates [17–20] are as follows

\[
\frac{B - B_m}{B_c - B_m} = V_c \left[ 1 + \left( 1 - V_c \right) \frac{3(B_c - B_m)}{3B_m + 4\mu_m} \right] \tag{1}
\]

\[
\frac{G - G_m}{G_c - G_m} = V_c \left[ 1 + \left( 1 - V_c \right) \frac{(G_c - G_m)}{G_m + f_1} \right] \tag{2}
\]

where,

\[ f_1 = \frac{G_m(9B_m + 8G_m)}{6(B_m + 2G_m)}. \]

Here, \( V \) is the volume fraction of the phase material. The subscripts \( c \) and \( m \) refer to the ceramic and metal phases, respectively. The volume-fractions of the ceramic and metal phases are related by \( V_c + V_m = 1 \), and \( V_c \) is expressed as

\[
V_c(z) = \left( \frac{2z + h}{2h} \right)^k \tag{3}
\]

where \( k \) is the volume fraction exponent \((k \geq 0)\).

The effective values of Young’s modulus \( E \) and Poisson’s ratio \( \nu \) can be found from \( E = \frac{9BG}{3B+2G} \) and \( \nu = \frac{3B-2G}{2(3B+G)} \), respectively.

The locally effective heat conductivity coefficient \( \kappa \) is given by [21]

\[
\frac{\kappa - \kappa_m}{\kappa_c - \kappa_m} = V_c \left[ 1 + \left( 1 - V_c \right) \frac{(\kappa_c - \kappa_m)}{3\kappa_m} \right]. \tag{4}
\]

The coefficient of thermal expansion \( \alpha \) is determined in terms of the correspondence relation [22]

\[
\frac{\alpha - \alpha_m}{\alpha_c - \alpha_m} = \frac{1}{K_c - \frac{1}{K_m}} \left( \frac{1}{K_c - \frac{1}{K_m}} \right). \tag{5}
\]

The effective mass density \( \rho \) can be given by the rule of mixtures as [23]

\[
\rho = \rho_c V_c + \rho_m V_m.
\]

The material properties \( P \) that are temperature dependent can be written as [28]

\[
P = P_0(1 - T^{-1} + P_1T + P_2T^2 + P_3T^3) \tag{6}
\]

where \( P_0, P_1, P_2 \) and \( P_3 \) are the coefficients of temperature \( T \) (K) and are unique to each constituent. Here, Young’s modulus and the thermal coefficient of expansion are considered as temperature dependent material properties.

The temperature variation is assumed to occur in the thickness direction only, and the temperature field is considered constant in the \( X–Y \) plane. Compared to the metal surface, the ceramic surface is exposed to higher temperatures. In this
mid-plane deformation of Eq. (9) for a plate as,
\begin{equation}
\{\varepsilon\} = \{\varepsilon^L\} + \{\varepsilon^{NL}\}.
\end{equation}

Taking into account the effect of shear deformation, the total linear and nonlinear strain at any point can be expressed as
\begin{equation}
\{\varepsilon^L\} = \left\{\begin{array}{c}
\varepsilon_{p}^L \\
0
\end{array}\right\} + \left\{\begin{array}{c}
\varepsilon_{s}^L \\
0
\end{array}\right\} \quad \text{and} \quad \{\varepsilon^{NL}\} = \left\{\begin{array}{c}
\varepsilon_{p}^{NL} \\
0
\end{array}\right\}.
\end{equation}

The mid-plane strains \{\varepsilon^L_p\}, bending strains \{\varepsilon_b\}, shear strains \{\varepsilon_s\}, and the nonlinear components of in-plane strains \{\varepsilon_p^{NL}\} in Eq. (11) are written as
\begin{equation}
\{\varepsilon^L_p\} = \left\{\begin{array}{c}
u_{o,x} \\
v_{o,y} \\
u_{o,y} + v_{o,x}
\end{array}\right\}
\end{equation}
\begin{equation}
\{\varepsilon_b\} = \left\{\begin{array}{c}
\theta_{x,x} \\
\theta_{y,y} \\
\theta_{x,x} + \theta_{y,y}
\end{array}\right\}
\end{equation}
\begin{equation}
\{\varepsilon_s\} = \left\{\begin{array}{c}
\theta_x + w_{o,x} \\
\theta_y + w_{o,y}
\end{array}\right\}
\end{equation}
\begin{equation}
\{\varepsilon_p^{NL}\} = \left\{\begin{array}{c}(1/2)w_{o,x}^2 \\
(1/2)w_{o,y}^2 \\
w_{o,x}w_{o,y}
\end{array}\right\}
\end{equation}
where the subscript comma denotes the partial derivative with respect to the spatial coordinate succeeding it.

The membrane stress resultants \{N\} and the bending stress resultants \{M\} can be related to the membrane strains \{\varepsilon_p\} (=\{\varepsilon^L_p\} + \{\varepsilon_p^{NL}\}) and bending strains \{\varepsilon_b\} through the constitutive relations by
\begin{equation}
\{N\} = \begin{bmatrix}
N_{xx} \\
N_{xy}
\end{bmatrix} = [A_{ij}][\varepsilon_p] + [B_{ij}][\varepsilon_b] - \{N^T\}
\end{equation}
\begin{equation}
\{M\} = \begin{bmatrix}
M_{xx} \\
M_{xy}
\end{bmatrix} = [B_{ij}][\varepsilon_b] + [D_{ij}][\varepsilon_b] - \{M^T\}
\end{equation}
where the matrices \{A_{ij}\}, \{B_{ij}\}, and \{D_{ij}\} (i, j = 1, 2, 6) are the extensional, bending–extensional coupling, and bending stiffness coefficients and are defined as \[ [A_{ij}, B_{ij}, D_{ij}] = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2) dz \]. The thermal stress resultant \{N^T\} and moment resultant \{M^T\} are
\begin{equation}
\{N^T\} = \begin{bmatrix}
N_{xx}^T \\
N_{xy}^T
\end{bmatrix} = \int_{-h/2}^{h/2} \left\{\begin{array}{c}
\alpha(z, T) \\
0
\end{array}\right\} \Delta T(z) dz
\end{equation}
\begin{equation}
\{M^T\} = \begin{bmatrix}
M_{xx}^T \\
M_{xy}^T
\end{bmatrix} = \int_{-h/2}^{h/2} \left\{\begin{array}{c}
\alpha(z, T) \\
0
\end{array}\right\} \Delta T(z) dz
\end{equation}
where the thermal coefficient of expansion \(\alpha(z, T)\) is given by Eq. (5), and \(\Delta T(z) = T(z) - T_0\) is the rise in temperature.
from the reference temperature \( T_0 \) at which there are no thermal strains. \( T_0 \) is assumed to be 300 K (27 °C).

Similarly, the transverse shear force \( (Q) \) representing the quantities \( \{Q_{1z}, Q_{2z}\} \) is related to the transverse shear strains \( \{\varepsilon_s\} \) through the constitutive relations as

\[
(\mathbf{Q}) = [E_{ij}] \{\varepsilon_s\}.
\]

where \( E_{ij} = \int_{-h/2}^{h/2} (\tilde{Q}_{ij}) k_i k_j dz \).

Here, \( [E_{ij}] \) (\( i, j = 4, 5 \)) are the transverse shear stiffness coefficients, \( k_i \) is the transverse shear coefficient for the non-uniform shear strain distribution through the plate thickness. \( \tilde{Q}_{ij} \) are the stiffness coefficients and are defined as

\[
\tilde{Q}_{11} = \tilde{Q}_{22} = \frac{E(z, T)}{1 - \nu^2}; \quad \tilde{Q}_{12} = \frac{\nu E(z, T)}{1 - \nu^2}; \quad \tilde{Q}_{16} = \tilde{Q}_{26} = 0; \quad \tilde{Q}_{44} = \tilde{Q}_{55} = \tilde{Q}_{66} = \frac{E(z, T)}{2(1 + \nu)}.
\]

The strain energy functional \( U \) is given as

\[
U(\delta) = (1/2) \int_A \left[ (\varepsilon_p)^T \{A_{ij}\} [\varepsilon_p] + (\varepsilon_b)^T \{B_{ij}\} [\varepsilon_b] + (\varepsilon_s)^T \{E_{ij}\} [\varepsilon_s] - (\varepsilon_p^0)^T \{N^T\} - (\varepsilon_b^T)^T \{M^T\} \right] dA
\]

where \( \delta \) is the vector of the degree of freedom associated to the displacement field in a finite element discretisation.

Following the standard procedure [24], the potential energy functional \( U \) given in Eq. (22) can be rewritten as

\[
U(\delta) = \delta^T \left( (1/2) [\mathbf{K}] + (1/6) [\mathbf{N}_1(\delta)] + (1/12) [\mathbf{N}_2(\delta)] + (1/2) [\mathbf{N}_3(\delta)] \right) \delta
\]

where \( [\mathbf{K}] \) is the linear stiffness matrix of the laminate, \( [\mathbf{N}_1] \) and \( [\mathbf{N}_2] \) are nonlinear stiffness matrices linearly and quadratically dependent on the field variables, respectively. \( [\mathbf{N}_3] \) is the transverse shear stiffness matrix of the plate.

The plate is subjected to a temperature field and this, in turn, results in in-plane stress resultants \( (N_{x1}^{th}, N_{y1}^{th}, N_{xy}^{th}) \). Thus, the potential energy due to pre-buckling stresses \( (N_{x1}^{th}, N_{y1}^{th}, N_{xy}^{th}) \) developed under thermal load can be written as

\[
V(\delta) = \int_A \left[ \frac{1}{2} \left( N_{x1}^{th} \left( \frac{\partial w}{\partial x} \right)^2 + N_{y1}^{th} \left( \frac{\partial w}{\partial y} \right)^2 \right) + \frac{h^3}{24} \left( N_{x1}^{th} \left( \frac{\partial \theta_x}{\partial x} \right)^2 + \left( \frac{\partial \theta_y}{\partial x} \right)^2 \right) + N_{xy}^{th} \left( \frac{\partial \theta_x}{\partial y} \right)^2 + \left( \frac{\partial \theta_y}{\partial y} \right)^2 \right] dA.
\]

By minimization of total potential energy obtained from Eqs. (23) and (24), the governing equation is derived as

\[
(\mathbf{K}) + (1/2) [\mathbf{N}_1] + (1/3) [\mathbf{N}_2] + [\mathbf{N}_3] \delta - \lambda [\mathbf{K}_G] \delta = \lambda (\mathbf{F}^T).
\]

This governing equation is obtained using the finite element approach. \( \mathbf{K}_G \) and \( \mathbf{K} \) are the geometric stiffness matrix due to initial thermal load \( (T_m) \) and the geometric stiffness matrix due to the unit temperature difference \( \Delta T \) \( (=T_e - T_m) \) respectively.

In the present work, an eight-noded \( C^0 \) continuous shear flexible plate bending element with five degrees of freedom per node is used to study the nonlinear behavior of FGM skew plates. By employing the field consistency approach described by Prathap et al. [15] and Ganapathi et al. [16], the element is found to be free from the locking syndrome and has good convergence properties. For skew plates, the element matrices corresponding to global axes are transformed to local axes using transformation rules [25].

3. Solution procedure

For the case of the pure ceramic or the metallic plate under uniform temperature rise, only membrane forces are generated. The critical buckling temperature difference at which an Euler type of buckling occurs is found from solving the following eigenvalue problem:

\[
(\mathbf{K}) + (1/2) [\mathbf{N}_1] + (1/3) [\mathbf{N}_2] + [\mathbf{N}_3] \delta - \lambda [\mathbf{K}_G] \delta = 0.
\]

Here, the force vector on the right-hand side is zero as the temperature effects (membrane forces) are accounted for in the geometric stiffness matrix. For the temperature dependent materials case, all the matrices given in Eq. (26) are functions of temperature. However, the nonlinear stiffness matrices depend on both the material properties and the displacement vectors. Firstly, using thermo-elastic properties at \( T_m \), the critical eigenvalue (critical buckling temperature difference, \( \Delta T_{cr} \)) and its associated eigenvector are obtained by neglecting the nonlinear stiffness matrices in Eq. (26). Next, considering the thermo-elastic properties at \( T = T_m + \Delta T_{cr} \), the linear matrices are updated, and the corresponding critical eigenvalue \( (\Delta T_{cr}) \) is extracted. This iterative procedure of accounting for temperature dependent properties is repeated until the critical buckling temperature difference \( (\Delta T_{cr}) \) converges to the desired accuracy.

For the nonlinear case, as mentioned above, all the matrices given in Eq. (26) are calculated using material properties at \( T = T_m + \Delta T_{cr} \). However, to evaluate the nonlinear part, in addition to the material properties at \( T = T_m + \Delta T_{cr} \), the displacement vector is required. To start with, the eigenvector obtained from the linear analysis is normalized and then scaled up to the desired amplitude, say \( w/h = 0.2 \) (\( w \) is the maximum...
lateral displacement, \( h \) is the thickness of the plate). Then it is
used for the evaluation of the nonlinear stiffness matrices. Subse-
quently, the updated eigenvalue (\( \Delta T_{\text{ex}} \)) and its corre-
sponding eigenvector are extracted for the specified amplitude. Next,
using the thermo-elastic properties at the updated temperature
(\( T = T_m + \Delta T_{\text{ex}} \)), and the scaling up of the normalized vec-
tor by the same amplitude (\( w/h \)), all the matrices are updated.
As part of the iteration, the critical eigenvalue and its eigenvector
are again obtained. This iterative procedure continues for
the chosen amplitude till the convergence of the buckling load
(eigenvalue) to the desired accuracy.

In the case of FGM plates under linear or nonlinear
temperature distributions through the thickness, bending
moments develop together with the membrane forces, which
are not included in the geometric stiffness matrix. Hence, the
force vector on the right-hand-side exists, and the eigenvalue
type of buckling may not be valid. In such cases, the nonlinear
equilibrium equation (25) is solved by the Newton–Raphson
technique to get the temperature–displacement curves.

4. Results and discussion

The study here has been focused on the nonlinear behavior
of functionally graded skew plates under constant, linear, and
nonlinear temperature variations through thickness. The plate
is of uniform thickness and is simply supported or clamped on
all four edges. The boundary conditions considered here are

(a) Simply supported
\[
\begin{align*}
\varphi_0 &= \nu_0 = \varphi = \theta_y = 0 \quad &\text{on} & \quad x = 0, a \\
\varphi_0 &= \nu = \varphi = \theta_x = 0 \quad &\text{on} & \quad y = 0, b
\end{align*}
\]

(b) Clamped support
\[
\begin{align*}
\varphi_0 &= \nu_0 = \varphi = \theta_x = \theta_y = 0 \quad &\text{on} & \quad x = 0, a \ &\text{&} & \quad y = 0, b.
\end{align*}
\]

In order to validate the efficacy of the present model for the
thermal postbuckling analysis of FGM plates, two examples are
considered for which solutions are available in the literature.
Firstly, the thermal buckling of thin clamped isotropic skew
plates under a uniform temperature rise has been analyzed
with different mesh sizes. The material properties (independent
of temperature), and the geometric parameters considered for
the numerical study are taken as \( E = 1.0 \) GPa, \( \nu = 0.3, \)
\( \alpha = 10^{-6} ^\circ \text{C} \), \( a = b \), and \( a/h = 100 \). The nondimensional
buckling temperature \( T^* = \Delta T \gamma h \) (evaluated based
on eigenvalue analysis (Eq. (26)), is given in Table 1 along
with those of Prabhu and Durvasula [26]. It is observed from
Table 1 that the convergence is monotonic, and the present
results compare well with the existing solutions. Further, based
on the progressive mesh refinement, an \( 8 \times 8 \) mesh is found
to be adequate to model the skew plate for the buckling
analysis. Secondly, the thermal postbuckling equilibrium path
of a uniformly heated isotropic clamped skew plate (skew angle
45°, \( a/b = 1 \), temperature independent material property)
obtained using the present formulation is highlighted in Fig. 2
and it is found to be in excellent agreement with the solution of
Prabhu and Durvasula [27].

Next, a detailed study of the thermal postbuckling of
aluminium/alumina FGM plates is carried out considering the
material properties as independent of temperature. The top
surface of the plate is ceramic rich and the bottom surface
is metal rich. The Young’s modulus, conductivity, and the
coefficient of thermal expansion for alumina are \( E_c = 380 \) GPa,
\( \kappa_c = 10.4 \) W/m K, \( \alpha_c = 7.4 \times 10^{-6} /\text{K} \), and for aluminium
are \( E_m = 70 \) GPa, \( \kappa_m = 204 \) W/m K, \( \alpha_m = 23 \times 10^{-6} /\text{K} \), respectively. Fig. 3(a) shows the variation of the

Fig. 2. Postbuckling equilibrium path of the isotropic clamped skew plate
(\( a/h = 100, a/b = 1, \) skew angle = 45°. Temperature independent material
property).

Fig. 3. Variation of the volume fraction of ceramic and the temperature
distribution through the plate thickness: (a) Volume fraction of ceramic; (b)
Temperature.
volume fraction of ceramic in the thickness direction ($z$) for the functionally graded skew plate. The typical temperature variation in the thickness direction (as per Eq. (8)) with different volume fraction indexes is presented in Fig. 3(b) assuming $\Delta T_c/\Delta T_m = 15$ and a temperature rise of the metal-rich surface of the plate as $\Delta T_m = 5 ^\circ$C ($T_m = 305$ K). It can be noted that the temperature variation through the thickness of the functionally graded plate is highly nonlinear compared to those of the pure ceramic and metal cases ($k = 0$ and $k = 100$). In addition to the nonlinear temperature distribution across the plate thickness, the linear case is also studied here for comparison purposes.

The thermal postbuckling characteristics of thin clamped aluminium/alumina FGM square ($a/h = 100$, $a/b = 1$, $T_m = 305$ K, temperature independent material property) plates under nonlinear temperature distribution.
It is observed that, due to the strong material property, the postbuckling temperature value increases monotonically with the increase in out-of-plane deformation, followed by a sudden drop in the postbuckling resistance (temperature). This sudden drop in the postbuckling resistance corresponds to a change in the mode shape (i.e. mode redistribution) as reported by Singha et al. [11] for composite laminates. After secondary instability, the postbuckling resistance increases marginally with the increase in out-of-plane deformation. Furthermore, the temperature–displacement curves are found to be symmetric with respect to the median surface for all the values of the material gradient index, i.e. irrespective of the direction of the buckling deformation. It is also opined that, with the increase in the material gradient index value \( k \), the resistance of the plate reduces, i.e., for a specified thermal load the out-of-plane displacement increases, and this is because of the stiffness reduction due to the higher metal inclusion in the FGM plate.

Next, the nonlinear bending behavior of simply supported thin square aluminium/alumina FGM plates \( (a/h = 100, a/b = 1, T_m = 305 \text{ K}, \text{temperature independent material property}) \) under a nonlinear through-thickness temperature distribution is investigated, and it is exhibited in Fig. 5 for various values of the material gradient index. It can be noted that the nonlinear governing equation \( (25) \) is solved by the Newton–Raphson technique to include the effects of extension–bending coupling and the non-symmetric-through-thickness thermal load (thermal moment \( \{M_T\} \)). The temperature–displacement relationships evaluated are shown by solid lines in Fig. 5. It is observed that, due to the strong bending moment \( \{M_T\} \) which develops in the FGM plate, the plate starts bending towards the upper side, where the ceramic content is high. For comparison purposes, a study is also made using the eigenvalue approach (Eq. (26)) by neglecting the effect of the bending moment \( \{M_T\} \), and the corresponding postbuckling paths are also highlighted in the figure. It is observed that the postbuckling curves are symmetric about the \( Y \)-axis for the pure ceramic \( (k = 0) \) and pure metal cases \( (k = 100) \), whereas it loses its symmetry and is shifted towards the upper side when the ceramic content increases \( (k = 0.5, \text{and } 2.0) \). This is attributed to the extension–bending coupling and a shift in the neutral surface towards the high stiff ceramic side of the plate. Furthermore, for the chosen deflection, the temperature load evaluated based on nonlinear analysis is, in general, low compared to those of the buckling study. It can be further noticed that the nonlinear characteristics obtained from two different approaches are quite different from each other. The difference between the solid curves (nonlinear analysis) and dotted curves (eigenvalue analysis) clearly indicate the effect of the thermal bending moment \( \{M_T\} \), which has been neglected in the earlier investigations [3–6,8]. Hence, it may be opined that the bifurcation type of instability cannot take place in an actual situation when the postbuckling response curve is not symmetric.

![Fig. 6. Thermal postbuckling paths of a simply supported thin aluminium/alumina FGM square plate under constant temperature rise through thickness \( (a/h = 100, a/b = 1, \text{temperature independent material property}) \).](image)

![Fig. 7. Thermal postbuckling paths of a simply supported aluminium/alumina FGM plate under constant, linear and nonlinear temperature rise through thickness \( (a/h = 100, a/b = 1, k = 0.5 \text{ respectively, temperature independent material property}) \).](image)

### Table 1
Thermal buckling parameter \( (\lambda^*) \) for a thin clamped isotropic skew plate

<table>
<thead>
<tr>
<th>Skew angle</th>
<th>Present</th>
<th>Prabh and Durvasula [26]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4 x 4</td>
<td>6 x 6</td>
</tr>
<tr>
<td>0°</td>
<td>3.8495</td>
<td>3.7374</td>
</tr>
<tr>
<td>15°</td>
<td>4.0949</td>
<td>3.9782</td>
</tr>
<tr>
<td>30°</td>
<td>4.9908</td>
<td>4.8417</td>
</tr>
<tr>
<td>45°</td>
<td>7.0738</td>
<td>7.0110</td>
</tr>
</tbody>
</table>

\[
\lambda^* = \frac{E a b^2 T_m}{\pi^2 D}
\]
The problem of simply supported aluminium/alumina FGM plates under a uniform temperature rise is solved, and the results are presented in Fig. 6 for different values of the material gradient index (k) considering the temperature independent material properties. It can be seen that the isotropic plates, under a uniform temperature rise, exhibit bifurcation buckling, and the corresponding postbuckling paths for pure ceramic and metal cases (k = 0, 100) are traced by eigenvalue analysis. For FGM plates (k = 0.5, and 2.0), the nonlinear temperature–displacement curves obtained through the Newton–Raphson method are shown in the figure. In Fig. 7, for comparison purposes, the nonlinear temperature–displacement curves evaluated for a simply supported aluminum/alumina FGM plate (k = 0.5) under constant, linear, and nonlinear temperature distributions are highlighted, considering temperature independent material properties. At a higher temperature level, the constant temperature distribution produces more deflection compared to those of linear and nonlinear temperature distributions.

Lastly, the effect of temperature dependent material properties on the thermal postbuckling characteristics of FGM skew plates are investigated considering FGM materials made of Silicon nitride (Si₃N₄) and stainless steel (SUS304). The temperature coefficients for the Young’s modulus (E) and...
Table 2
Temperature dependent coefficients for the material Si$_3$N$_4$/SUS304, Ref. [28]

<table>
<thead>
<tr>
<th>Materials</th>
<th>Properties</th>
<th>$P_0$</th>
<th>$P_{-1}$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P$ ($T = 300$ K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Si$_3$N$_4$</td>
<td>$E$ (Pa)</td>
<td>348.43e+9</td>
<td>0.0</td>
<td>-3.070e-4</td>
<td>2.160e-7</td>
<td>-8.946e-11</td>
<td>322.2715e+9</td>
</tr>
<tr>
<td></td>
<td>$\alpha$ (1/K)</td>
<td>5.8723e-6</td>
<td>0.0</td>
<td>9.095e-4</td>
<td>0.0</td>
<td>0.0</td>
<td>7.4746e-6</td>
</tr>
<tr>
<td>SUS304</td>
<td>$E$ (Pa)</td>
<td>201.04e+9</td>
<td>0.0</td>
<td>3.079e-4</td>
<td>-6.534e-7</td>
<td>0.0</td>
<td>207.7877e+9</td>
</tr>
<tr>
<td></td>
<td>$\alpha$ (1/K)</td>
<td>12.330e-6</td>
<td>0.0</td>
<td>8.086e-4</td>
<td>0.0</td>
<td>0.0</td>
<td>15.321e-6</td>
</tr>
</tbody>
</table>

The coefficient of thermal expansion ($\alpha$) corresponding to Si$_3$N$_4$/SUS304 are listed in Table 2 [28]. The material properties ($E$ and $\alpha$) are evaluated at elevated temperatures using Eq. (6). The mass density and thermal conductivity are: $\rho_e = 2370$ kg/m$^3$, $\kappa_e = 9.19$ W/m K for Si$_3$N$_4$; and $\rho_m = 8166$ kg/m$^3$, $\kappa_m = 12.04$ W/m K for SUS304. The Poisson’s ratio $\nu$ is assumed to be a constant and equals 0.28.

Fig. 8 shows the thermal postbuckling paths for a simply supported Si$_3$N$_4$/SUS304 FGM thick plate ($a/h = 20$) for two values of the material gradient index ($k = 2$, 100). For the purpose of comparison, results considering material properties independent of temperature are also presented. From the figure, it is observed that the material degradation at a higher temperature increases the out-of-plane deflection ($w/h$) compared to those of the temperature independent material case. Hence, the thermal postbuckling resistance is overestimated when temperature independent material properties are used, as expected.

Lastly, the effect of the skew angle on the thermal postbuckling paths of simply supported thin Si$_3$N$_4$/SUS304 FGM skew plates ($a/h = 100$) under nonlinear through-the-thickness temperature variations is studied based on a nonlinear analysis, and the results are presented in Fig. 9. The temperature variations obtained for different skew angles are qualitatively similar with respect to ones for deflection. It is further observed that with an increase in the skew angle, the out-of-plane deflection ($w/h$) reduces for a specified thermal load, i.e., the load carrying capacity increases with the increase in the skew angle. Similarly, the thermal postbuckling paths of simply supported thick Si$_3$N$_4$/SUS304 FGM skew plates ($a/h = 20$) under nonlinear through-the-thickness temperature variations are also shown in Fig. 10. Although similar behavior is observed compared to those of the thin skew plates, the difference in the thermal postbuckling resistance between the temperature dependent case and the temperature independent material property case is significantly high for the thick situation as compared to those of thin skew plates.

5. Conclusions

The nonlinear behavior of FGM skew plates under a thermal load is studied here using an eight-noded shear flexible plate bending finite element. FGM materials made of aluminium/alumina with temperature independent material properties, and Silicon nitride (Si$_3$N$_4$)/stainless steel (SUS304) with temperature dependent material properties, are considered in the numerical analysis. The nonlinear governing equations based on von Karman’s assumptions are solved with the Newton–Raphson method to study the nonlinear behavior of FGM plates under thermal loads. From the detailed parametric study, some of the observations made are as follows:

- FGM plates with clamped boundary conditions exhibit a bifurcation type of instability under thermal load and the corresponding postbuckling path can be obtained from the eigenvalue analysis.
- FGM plates with a simply supported boundary condition requires nonlinear analysis as the plates exhibit bending behavior due to the existence of extension–bending coupling and the thermal bending moment ($M_T$).
- The temperature resistance of FGM plates is highly dependent on the variation of the temperature distribution through the thickness of the plate.
An increase in thickness enhances the critical buckling as well as postbuckling resistance.

The temperature independent material property overestimates the thermal postbuckling resistance.

The thermal load carrying capacity increases with the increase in the skew angle.

The difference in the thermal postbuckling resistance between temperature dependent and temperature independent material property cases is significantly high for the thick plates as compared to those of the thin plates.

References


Fig. 10. Thermal postbuckling path of simply supported thin Si₃N₄/SUS304 FGM plates with different skew angles (a/h = 20, a/b = 1, T₀ = 305 K): (a) k = 0.5, (b) k = 2.0.

